Geophys. J. Int. (2010) **181,** 229–249 doi: 10.1111/j.1365-246X.2009.04470.x

General formalism for the efficient calculation of derivatives of EM frequency-domain responses and derivatives of the misfit

Oleg Pankratov[1] and Alexei Kuvshinov[2]

[1] *Institute of Terrestrial Magnetism, Ionosphere and Radiowave Propagation, Russian Academy of Sciences,* 142190, *Troitsk, Moscow region, Russia. E-mail:oleg.pankratov@gmail.com*
[2] *Institute of Geophysics, ETH,* CH-8093 *Zurich, Switzerland. E-mail: kuvshinov@erdw.ethz.ch*

Accepted 2009 December 2. Received 2009 September 4; in original form 2009 May 8

SUMMARY

Electromagnetic (EM) studies of the Earth have advanced significantly over the past few years. This progress was driven, in particular, by new developments in the methods of 3-D inversion of EM data. Due to the large scale of the 3-D EM inverse problems, iterative gradient-type methods have mostly been employed. In these methods one has to calculate multiple times the gradient of the penalty function—a sum of misfit and regularization terms—with respect to the model parameters. However, even with modern computational capabilities the straightforward calculation of the misfit gradients based on numerical differentiation is extremely time consuming. Much more efficient and elegant way to calculate the gradient of the misfit is provided by the so-called 'adjoint' approach. This is now widely used in many 3-D numerical schemes for inverting EM data of different types and origin. It allows the calculation of the misfit gradient for the price of only a few additional forward calculations. In spite of its popularity we did not find in the literature any general description of the approach, which would allow researchers to apply this methodology in a straightforward manner to their scenario of interest. In the paper, we present formalism for the efficient calculation of the derivatives of EM frequency-domain responses and the derivatives of the misfit with respect to variations of 3-D isotropic/anisotropic conductivity. The approach is rather general; it works with single-site responses, multisite responses and responses that include spatial derivatives of EM field. The formalism also allows for various types of parametrization of the 3-D conductivity distribution. Using this methodology one can readily obtain appropriate formulae for the specific sounding methods. To illustrate the concept we provide such formulae for a number of EM techniques: geomagnetic depth sounding (GDS), conventional and generalized magnetotellurics, the magnetovariational method, horizontal gradient sounding (HGS) and a method that combines HGS with GDS. We also show how the developed formalism can be adapted for the inversion of multisite responses—horizontal magnetic and electric tensors.

Key words: Inverse theory; Electromagnetic theory; Magnetotelluric; Geomagnetic induction.

1 INTRODUCTION

Electromagnetic (EM) studies of the conducting Earth, from the near surface to regional and global, have advanced significantly over the past few years. This progress has been achieved by the increased accuracy, coverage and variety of the newly available data sets, as well as by the new developments in the methods of 3-D modelling and 3-D inversion of EM data. Due to the large scale of the 3-D inverse problems iterative gradient-type methods (*cf*. Nocedal & Wright 2006) are believed to be the only feasible methods. These rely on multiple calculations of the gradient of the misfit function with respect to the model parameters. However, even with modern computational capabilities the straightforward calculation of the gradients based on numerical differentiation is forbidden due to the tremendous computational loads. A much more efficient and elegant way to calculate the misfit gradient is provided by so-called 'adjoint' approach (*cf*. Mackie & Madden 1993; Pellerin *et al.* 1993; McGillivray *et al.* 1994; among others) which is now widely used in many 3-D numerical schemes for inverting EM data of different types and origin, both in frequency (*cf*. Dorn *et al.* 1999; Newman & Alumbaugh 2000; Rodi & Mackie 2000; Haber 2005; Kelbert *et al.* 2008; Avdeev & Avdeeva 2009; among others) and time (*cf*. Newman & Commer 2005; Haber *et al.* 2007, among others) domains. This approach allows one to calculate the misfit gradient for the price of only a few additional forward calculations. In spite of the growing popularity of the approach we did not find in the literature a comprehensive and general description, which would allow researchers to apply the adjoint methodology for their chosen EM problem in a straightforward manner. In this paper we try to fill this gap and present a general formalism for calculating sensitivities (derivatives) of the response functions and gradients of the misfit. We restrict our task by considering only the frequency-domain formulation.

The paper is organized as follows. Section 2 introduces the operators of the 3-D EM forward problems and gives definitions of polarizations, sites, response functions and model parametrization. Section 3 describes the inverse problem formulation. Sections 4 and 5 present the general formalism for the calculation of derivatives of, respectively, the response function and the misfit. The concept is explained using an example of single-site responses. Section 6 demonstrates the application of the formalism to the responses of the following methods: geomagnetic depth sounding (GDS), conventional and generalized (*cf*. Dmitriev & Berdichevsky 2002) magnetotelluric (MT) methods and magnetovariational (MV) method. In Section 6, we also show how the formalism can be adapted to the inversion of multisite responses (including horizontal magnetic and electric tensors), and to the inversion of the responses that contain spatial derivatives of EM field (arising in horizontal gradient sounding (HGS) method, and in a method that combines HGS with GDS (*cf*. Schmucker 2003; Semenov *et al.* 2007)). Section 7 presents a generalization of the approach to the case of anisotropic conductivity, alternative model parametrizations, and an application to non-holomorphic responses. Conclusions are drawn in Section 8. This paper also includes six Appendices. Appendix A demonstrates reciprocity of Maxwell's equations' solutions—a property that we use in Section 4. Appendix B introduces a representation of the response functions as a coupling matrix. Such a representation allows the calculation of the derivatives of response functions in closed form, which are presented in Appendix C. The results obtained in Appendices B and C are exploited throughout Section 6. In Appendix D, we derive and explain a formula for the differential of the quadratic type complex-valued function. Appendices E and F present respective extensions of the formalism on the case of multisite responses, and responses that contain spatial derivatives of EM field. The results obtained in Appendices E and F are exploited in Sections 6.5–6.8.

2 DEFINITIONS

2.1 Operators $\mathbf{G}^e$, $\mathbf{G}^{ej}$, $\mathbf{G}^{hj}$ and $\mathbf{G}^{eh}$

Let us define an operator $\mathbf{G}^e$ in the whole Euclidean space $\mathbb{R}^3$ as follows:

$$\mathbf{E} = \mathbf{G}^e\left(\mathbf{j}^{imp}, \mathbf{h}^{imp}\right) \quad \Leftrightarrow \quad \begin{cases} \nabla \times \mathbf{H} = \sigma \mathbf{E} + \mathbf{j}^{imp}, \\ \nabla \times \ \mathbf{E} = i\omega\mu\mathbf{H} + \mathbf{h}^{imp}, \\ \mathbf{E}(\mathbf{r}), \mathbf{H}(\mathbf{r}) \to 0 \ \text{ as } \ \mathbf{r} \to \infty, \end{cases} \tag{1}$$

where $\mathbf{j}^{imp}(\mathbf{r})$ and $\mathbf{h}^{imp}(\mathbf{r})$ are impressed (extraneous) electric current and magnetic dipoles distribution, $\mathbf{r} \in \mathbb{R}^3$, $i = \sqrt{-1}$, $\sigma(\mathbf{r})$ and $\mu(\mathbf{r})$ are electric conductivity and magnetic permeability distributions in an earth model, respectively. Time dependence is accounted for by $e^{-i\omega t}$ with $\omega = 2\pi/T$, where T is the period. (Note, that for the case when $\mathrm{Im}\sigma \neq 0$, we will apply the radiation condition at the infinity, instead of the condition $\mathbf{E}(\mathbf{r}),\ \mathbf{H}(\mathbf{r}) \to 0$ as $\mathbf{r} \to \infty$.) Operator $\mathbf{G}^e(\cdot)$ acts on $(\mathbf{j}^{imp}, \mathbf{h}^{imp})$ and produces electric field $\mathbf{E} = \mathbf{G}^e(\mathbf{j}^{imp}, \mathbf{h}^{imp})$. At this stage we don't specify the coordinate system in the Euclidean space $\mathbb{R}^3$; this means that $\mathbf{r} = \mathbf{r}(\chi, \gamma, \zeta)$ can be, say, Cartesian coordinates, (x, y, z) or spherical coordinates, (r, ϑ, φ). Next we define an operator $\mathbf{G}^{ej}$ as a restriction of operator $\mathbf{G}^e$ to electric sources only, $(\mathbf{j}^{imp}, \mathbf{h}^{imp}) = (\mathbf{j}^{imp}, 0)$

$$\mathbf{E} = \mathbf{G}^{ej}(\mathbf{j}^{imp}) \ \equiv \mathbf{G}^e(\mathbf{j}^{imp}, 0) \quad \Leftrightarrow \quad \begin{cases} \nabla \times \mathbf{H} = \sigma \mathbf{E} + \mathbf{j}^{imp}, \\ \nabla \times \ \mathbf{E} = i\omega\mu\mathbf{H}, \\ \mathbf{E}(\mathbf{r}), \mathbf{H}(\mathbf{r}) \to 0 \ \text{ as } \ \mathbf{r} \to \infty. \end{cases} \tag{2}$$

Operator $\mathbf{G}^{ej}(\cdot)$ acts on $\mathbf{j}^{imp}$ and produces electric field $\mathbf{E} = \mathbf{G}^{ej}(\mathbf{j}^{imp})$. We also define operator $\mathbf{G}^{eh}$, which is a restriction of operator $\mathbf{G}^e$ to magnetic sources only, $(\mathbf{j}^{imp}, \mathbf{h}^{imp}) = (0, \mathbf{h}^{imp})$

$$\mathbf{E} = \mathbf{G}^{eh}(\mathbf{h}^{imp}) \equiv \mathbf{G}^e(0, \mathbf{h}^{imp}) \quad \Leftrightarrow \quad \begin{cases} \nabla \times \mathbf{H} = \sigma \mathbf{E}, \\ \nabla \times \ \mathbf{E} = i\omega\mu\mathbf{H} + \mathbf{h}^{imp}, \\ \mathbf{E}(\mathbf{r}), \mathbf{H}(\mathbf{r}) \to 0 \ \text{ as } \ \mathbf{r} \to \infty. \end{cases} \tag{3}$$

Operator $\mathbf{G}^{eh}(\cdot)$ acts on $\mathbf{h}^{imp}$ and produces electric field $\mathbf{E} = \mathbf{G}^{eh}(\mathbf{h}^{imp})$. It can be shown that operators $\mathbf{G}^{eh}$ and $\mathbf{G}^{ej}$ are related by

$$\mathbf{G}^{eh}(\mathbf{h}^{imp}) = \mathbf{G}^{ej}\left(\nabla \times \frac{\mathbf{h}^{imp}}{i\omega\mu}\right), \tag{4}$$

for any distribution of extraneous magnetic dipoles $\mathbf{h}^{imp}$. Due to linearity of eqs (1)–(3) with respect to the source, the operator $\mathbf{G}^e$ can be expressed via operators $\mathbf{G}^{ej}$ and $\mathbf{G}^{eh}$ as

$$\mathbf{G}^e(\mathbf{j}^{imp}, \mathbf{h}^{imp}) = \mathbf{G}^e(\mathbf{j}^{imp}, 0) + \mathbf{G}^e\left(0, \mathbf{h}^{imp}\right) = \mathbf{G}^{ej}(\mathbf{j}^{imp}) + \mathbf{G}^{eh}(\mathbf{h}^{imp}). \tag{5}$$

In a similar way we define operator $\mathbf{G}^{hj}$

$$\mathbf{H} = \mathbf{G}^{hj}(\mathbf{j}^{imp}) \quad \Leftrightarrow \quad \begin{cases} \nabla \times \mathbf{H} = \sigma \mathbf{E} + \mathbf{j}^{imp}, \\ \nabla \times \ \mathbf{E} = i\omega\mu\mathbf{H}, \\ \mathbf{E}(\mathbf{r}), \mathbf{H}(\mathbf{r}) \to 0 \ \text{as} \ \mathbf{r} \to \infty, \end{cases} \tag{6}$$

which acts on $\mathbf{j}^{imp}$ and produces a magnetic field $\mathbf{H} = \mathbf{G}^{hj}(\mathbf{j}^{imp})$. Operators $\mathbf{G}^{eh}$ and $\mathbf{G}^{hj}$ are related by eq. (A2). We assume that operators $\mathbf{G}^{e}$, $\mathbf{G}^{ej}$, $\mathbf{G}^{hj}$ and $\mathbf{G}^{eh}$ are in our possession—namely, we can numerically solve Maxwell's eqs (1)–(3) using, for example, finite-difference, finite-element or integral equation approaches.

2.2 Polarizations

Let

$$\{\mathbf{j}_p\}_{p \in Polars}, \quad Polars = \{1, 2, \ldots, N_P\}, \tag{7}$$

be a set of linearly independent distributions of the impressed electric currents, $\mathbf{j}_p = \mathbf{j}_p(\mathbf{r})$, $p \in Polars$. For example, in MT studies $N_P = 2$, and $\mathbf{j}_1$ and $\mathbf{j}_2$ correspond to the plane waves of different orientations. Each $\mathbf{j}_p$ produces electric, $\mathbf{E}_p$ and magnetic, $\mathbf{H}_p$, fields that can be written via operators $\mathbf{G}^{ej}$ and $\mathbf{G}^{hj}$ as

$$\begin{cases} \mathbf{E}_p = \mathbf{G}^{ej}(\mathbf{j}_p), \\ \mathbf{H}_p = \mathbf{G}^{hj}(\mathbf{j}_p). \end{cases} \tag{8}$$

Note that we deliberately consider the formulation for electric sources only since the formulation for magnetic sources can be reduced to the formulation for electric sources using eq. (4).

2.3 Inversion domain and model parametrization

Let $V^{inv} \subseteq \mathbb{R}^3$ be the inversion domain where we seek the conductivity distribution. Let

$$\{V_m\}_{m \in Model}, \qquad Model = \{1, \ldots, N_M\}, \tag{9}$$

be a set of elementary volumes V_m, where all V_m comprise the inversion domain

$$\bigcup_{m=1}^{N_M} V_m = V^{inv}, \tag{10}$$

and within each volume V_m the conductivity is constant

$$\sigma(\mathbf{r}) = \sigma_m, \quad \text{for} \quad \mathbf{r} \in V_m. \tag{11}$$

Then the vector

$$\mathbf{m} = (\sigma_1, \sigma_2, \ldots, \sigma_{N_M}), \tag{12}$$

defines the model parametrization that we use throughout most of the paper. Note that some of these elementary volumes might be cells (or combinations of cells) of 3-D part of the model, whereas others might be the layers of 1-D part of the model, provided that eq. (10) fulfils and $V_m \cap V_n = \emptyset$ for any two different volumes $V_m, \ V_n, \ m, n \in Model$.

2.4 Frequencies, observation sites and response functions

Let

$$\Omega = \{\omega_i\}_{i=1}^{N_\Omega}, \tag{13}$$

$$\{\mathbf{r}_a\}_{a \in Sites}, \quad Sites = \{1, \ldots, N_S\}, \tag{14}$$

$$\{\Phi^{(k)}\}_{k \in Resps}, \quad Resps = \{1, \ldots, N_\Phi\}, \tag{15}$$

be sets of frequencies, ω_i, observation sites, $\mathbf{r}_a \in \mathbb{R}^3$ and complex-valued response functions, $\Phi^{(k)}$, respectively. If we fix site $\mathbf{r}_a$ and frequency ω_i then the response functions depend on electric and/or magnetic fields of different polarizations

$$\Phi_a^{(k)} \equiv \Phi^{(k)}|_{\mathbf{r}_a} = \Phi^{(k)}\left(\mathbf{E}_1(\mathbf{r}_a), \ldots, \mathbf{E}_{N_p}(\mathbf{r}_a), \mathbf{H}_1(\mathbf{r}_a), \ldots, \mathbf{H}_{N_p}(\mathbf{r}_a)\right). \tag{16}$$

For example, in the conventional magnetotelluric (MT) method the customary response functions are the elements of the impedance tensor, $\Phi^{(1)} = Z_{xx}, \ \Phi^{(2)} = Z_{xy}, \ \Phi^{(3)} = Z_{yx}, \ \Phi^{(4)} = Z_{yy}$ where

$$\begin{cases} E_x = Z_{xx} H_x + Z_{xy} H_y, \\ E_y = Z_{yx} H_x + Z_{yy} H_y. \end{cases} \tag{17}$$

In geomagnetic deep sounding (GDS) the response function is the so-called local *Z:H C*-response (*cf*. Banks 1969)

$$\Phi^{(1)} = C = -\frac{R_e}{2}\tan\vartheta_d \frac{Z}{H}. \tag{18}$$

Here R_e is the mean radius of the Earth, ϑ_d is the geomagnetic colatitude, and Z and H are, respectively, the vertical and horizontal (directed towards geomagnetic north) components of the magnetic field.

3 INVERSE PROBLEM FORMULATION

We formulate the inverse problem of conductivity recovery as an optimization problem such that

$$\phi(\mathbf{m}, \lambda) \underset{\mathbf{m}}{\rightarrow} \min, \tag{19}$$

with a penalty function

$$\phi(\mathbf{m}, \lambda) = \phi_d(\mathbf{m}) + \lambda\phi_s(\mathbf{m}), \tag{20}$$

where λ and $\phi_s(\mathbf{m})$ are a regularization parameter and stabilizer, respectively, and $\phi_d(\mathbf{m})$ is the data misfit

$$\phi_d(\mathbf{m}) = \sum_{k\in Resps}\sum_{\omega\in\Omega}\sum_{a\in Sites} D_a^{(k)}(\omega)\cdot\left|\Phi_a^{(k)}(\mathbf{m},\omega) - \Phi_a^{(k),\,exp}(\omega)\right|^2. \tag{21}$$

Here $\Phi_a^{(k)}(\mathbf{m},\omega)$ and $\Phi_a^{(k),\,exp}(\omega)$ are, respectively, the predicted and observed values of the response functions at observation site $\mathbf{r}_a$ and frequency ω, and $D_a^{(k)}(\omega)$ are the inverses of squared uncertainties of the observed responses. Due to the large scale of the 3-D EM inverse problems we will solve the optimization problem (19)–(21) using gradient-type methods. In these methods one has to calculate the gradient of the penalty function. In what follows, we present general formalism to efficiently calculate the gradient of misfit. Note, that usually the evaluation of gradient of the regularization term is rather straightforward.

4 DERIVATIVES OF THE RESPONSE FUNCTION

Let us derive the differential, $d\Phi|_{r_a}$, with respect to variation of σ. In this section we will omit for simplicity the superscript '(k)' in Φ. According to the differentiation of a function, which in turn depends on other functions (see eq. 16) we have

$$d\Phi|_{\mathbf{r}_a} = \sum_{p\in Polars}\left(d_{\mathbf{E}_p}\Phi + d_{\mathbf{H}_p}\Phi\right)\Big|_{\mathbf{r}_a}, \tag{22}$$

where $d_{\mathbf{E}_p}\Phi$ and $d_{\mathbf{H}_p}\Phi$ are differentials of Φ with respect to respective variations of $\mathbf{E}_p$ and $\mathbf{H}_p$ for polarization $p \in Polars$. We can express the differential $d_{\mathbf{E}_p}\Phi$ in any coordinates (χ, γ, ζ) as

$$d_{\mathbf{E}_p}\Phi = \frac{\partial\Phi}{\partial\mathbf{E}_p}\cdot d\mathbf{E}_p = \frac{\partial\Phi}{\partial E_{\chi p}}dE_{\chi p} + \frac{\partial\Phi}{\partial E_{\gamma p}}dE_{\gamma p} + \frac{\partial\Phi}{\partial E_{\zeta p}}dE_{\zeta p}. \tag{23}$$

This can be thought as a product of a row matrix

$$\frac{\partial\Phi}{\partial\mathbf{E}_p} = \left(\frac{\partial\Phi}{\partial E_{\chi p}},\ \frac{\partial\Phi}{\partial E_{\gamma p}},\ \frac{\partial\Phi}{\partial E_{\zeta p}}\right), \tag{24}$$

and a column matrix

$$d\mathbf{E}_p = (dE_{\chi p},\ \ dE_{\gamma p},\ \ dE_{\zeta p})^T. \tag{25}$$

Here the superscript 'T' stands for the matrix transposition. In a similar way $d_{\mathbf{H}_p}\Phi$, $\partial\Phi/\partial\mathbf{H}_p$ and $d\mathbf{H}_p$ stand for

$$d_{\mathbf{H}_p}\Phi = \frac{\partial\Phi}{\partial\mathbf{H}_p}\cdot d\mathbf{H}_p = \frac{\partial\Phi}{\partial H_{\chi p}}dH_{\chi p} + \frac{\partial\Phi}{\partial H_{\gamma p}}dH_{\gamma p} + \frac{\partial\Phi}{\partial H_{\zeta p}}dH_{\zeta p}, \tag{26}$$

$$\frac{\partial\Phi}{\partial\mathbf{H}_p} = \left(\frac{\partial\Phi}{\partial H_{\chi p}},\ \frac{\partial\Phi}{\partial H_{\gamma p}},\ \frac{\partial\Phi}{\partial H_{\zeta p}}\right), \tag{27}$$

$$d\mathbf{H}_p = (dH_{\chi p},\ \ dH_{\gamma p},\ \ dH_{\zeta p})^T. \tag{28}$$

We will now demonstrate the key result of this study that

$$d_{\mathbf{E}_p}\Phi|_{\mathbf{r}_a} = \left(\frac{\partial\Phi}{\partial\mathbf{E}_p}\cdot d\mathbf{E}_p\right)\Big|_{\mathbf{r}_a} = \left\langle \mathbf{G}^{ej}\left(\frac{\partial\Phi}{\partial\mathbf{E}_p}\delta_{\mathbf{r}_a}\right),\quad d\sigma\ \mathbf{G}^{ej}(\mathbf{j}_p)\right\rangle, \tag{29}$$

$$d_{\mathbf{H}_p}\Phi|_{\mathbf{r}_a} = \left(\frac{\partial\Phi}{\partial\mathbf{H}_p}\cdot d\mathbf{H}_p\right)\Big|_{\mathbf{r}_a} = \left\langle \mathbf{G}^{eh}\left(\frac{\partial\Phi}{\partial\mathbf{H}_p}\delta_{\mathbf{r}_a}\right),\quad d\sigma\ \mathbf{G}^{ej}(\mathbf{j}_p)\right\rangle, \tag{30}$$

where $\delta_{\mathbf{r}_a} = \delta(\mathbf{r} - \mathbf{r}_a)$ is Dirac's delta function, and angle brackets, $\langle\cdot,\cdot\rangle$, denote

$$\langle\mathbf{F},\mathbf{G}\rangle = \int_{\mathbb{R}^3}\mathbf{F}(\mathbf{r})\cdot\mathbf{G}(\mathbf{r})dv(\mathbf{r}), \tag{31}$$

where $\mathbf{F}(\mathbf{r}) \cdot \mathbf{G}(\mathbf{r}) = F_\chi(\mathbf{r})G_\chi(\mathbf{r}) + F_\gamma(\mathbf{r})G_\gamma(\mathbf{r}) + F_\varsigma(\mathbf{r})G_\varsigma(\mathbf{r})$ is a complex-valued bilinear scalar product, and $dv(\mathbf{r})$ is an elementary volume.

Let us first prove that differentials of electric and magnetic fields with respect to variation of conductivity can be written in the form (*cf*. Zhdanov 2002)

$$d\mathbf{E}_p = \mathbf{G}^{ej}\left(d\sigma\ \mathbf{G}^{ej}(\mathbf{j}_p)\right), \tag{32}$$

$$d\mathbf{H}_p = \mathbf{G}^{hj}\left(d\sigma\ \mathbf{G}^{ej}(\mathbf{j}_p)\right). \tag{33}$$

Proof of eq. (32)**.** Let us consider EM field ($\mathbf{E} \equiv \mathbf{E}_p, \mathbf{H} \equiv \mathbf{H}_p$) that satisfies eq. (2) with $\mathbf{j}^{imp}$ being equal to $\mathbf{j}_p$. Let us also consider the EM field ($\mathbf{E} + \Delta\mathbf{E}, \mathbf{H} + \Delta\mathbf{H}$) that satisfies Maxwell's equations for $\sigma + \Delta\sigma$:

$$\begin{cases} \nabla \times\ (\mathbf{H} + \Delta\mathbf{H}) = (\sigma + \Delta\sigma)(\mathbf{E} + \Delta\mathbf{E}) + \mathbf{j}_p, \\ \quad \nabla \times\ (\mathbf{E} + \Delta\mathbf{E}) = i\omega\mu(\mathbf{H} + \Delta\mathbf{H}). \end{cases} \tag{34}$$

We notice from eq. (2) that $\mathbf{E} = \mathbf{G}^{ej}(\mathbf{j}_p)$. Subtracting eq. (2) from eq. (34) we get

$$\begin{cases} \nabla \times \Delta\mathbf{H} = (\sigma + \Delta\sigma)\Delta\mathbf{E} + \Delta\sigma\ \mathbf{G}^{ej}(\mathbf{j}_p), \\ \qquad \nabla \times \Delta\mathbf{E} = i\omega\mu\Delta\mathbf{H}. \end{cases} \tag{35}$$

In the limit of $\Delta\sigma \to 0$ and ignoring the second order quantity $\Delta\sigma\ \Delta\mathbf{E}$, we arrive at the following equations

$$\begin{cases} \nabla \times d\mathbf{H} = \sigma\ d\mathbf{E} + d\sigma\ \mathbf{G}^{ej}(\mathbf{j}_p), \\ \quad \nabla \times d\mathbf{E} = i\omega\mu\ d\mathbf{H}. \end{cases} \tag{36}$$

Thus, $d\mathbf{E} \equiv d\mathbf{E}_p = \mathbf{G}^{ej}(d\sigma\ \mathbf{G}^{ej}(\mathbf{j}_p))$ by definition (2). Comparing eq. (36) with eq. (6) we obtain eq. (33). It should be noted that eqs (32) and (33) are valid for both isotropic and anisotropic conductivity $\sigma(\mathbf{r})$; the latter is discussed in Section 7.1.

Proof of eq. (29). For simplicity, but without loss of generality, we will consider Cartesian coordinate system. Let us start by estimating the component $(\frac{\partial\Phi}{\partial E_{xp}} dE_{xp})|_{\mathbf{r}_a}$

$$\left(\frac{\partial\Phi}{\partial E_{xp}} dE_{xp}\right)\Big|_{\mathbf{r}_a} = \left(\frac{\partial\Phi}{\partial E_{xp}}\left[\mathbf{G}^{ej}\left(d\sigma\ \mathbf{G}^{ej}(\mathbf{j}_p)\right)\right]_x\right)\Big|_{\mathbf{r}_a} = \left\langle \frac{\partial\Phi}{\partial E_{xp}}\mathbf{e}_x\delta_a,\ \ \mathbf{G}^{ej}\left(d\sigma\ \mathbf{G}^{ej}(\mathbf{j}_p)\right)\right\rangle. \tag{37}$$

Here we used eq. (32). Using reciprocity of Green's function $\mathbf{G}^{ej}$ (see Appendix A, eq. A1) we have

$$\left(\frac{\partial\Phi}{\partial E_{xp}} dE_{xp}\right)\Big|_{\mathbf{r}_a} = \left\langle \frac{\partial\Phi}{\partial E_{xp}}\mathbf{e}_x\delta_a,\ \ \mathbf{G}^{ej}\left(d\sigma\ \mathbf{G}^{ej}(\mathbf{j}_p)\right)\right\rangle = \left\langle \mathbf{G}^{ej}\left(\frac{\partial\Phi}{\partial E_{xp}}\mathbf{e}_x\delta_a\right),\ \ d\sigma\ \mathbf{G}^{ej}(\mathbf{j}_p)\right\rangle. \tag{38}$$

Similar calculations may be performed for $(\frac{\partial\Phi}{\partial E_{yp}} dE_{yp})|_{\mathbf{r}_a}$ and $(\frac{\partial\Phi}{\partial E_{zp}} dE_{zp})|_{\mathbf{r}_a}$. Due to linearity of $\mathbf{G}^{ej}$ with respect to the source, summation of the three terms gives us the desired eq. (29).

Proof of eq. (30) follows from similar calculations and from eqs (33) and (A2). Combining eqs (22), (29), (30), and using eq. (5) we further have

$$d\Phi|_{\mathbf{r}_a} = \sum_{p \in Polars}\left\langle \mathbf{G}^{e}\left(\left(\frac{\partial\Phi}{\partial\mathbf{E}_p}, \frac{\partial\Phi}{\partial\mathbf{H}_p}\right)\delta_{\mathbf{r}_a}\right),\ \ d\sigma\ \mathbf{G}^{ej}(\mathbf{j}_p)\right\rangle. \tag{39}$$

Bearing in mind our model parametrization (see Section 2.3) we finally arrive at the following expressions for differentials, $d\Phi^{(k)}|_{\mathbf{r}_a}$, with respect to variation of current model conductivities σ_m

$$d\Phi^{(k)}|_{\mathbf{r}_a} = \sum_{m \in Model} \kappa_m^{(k)}(\mathbf{r}_a) d\sigma_m, \tag{40}$$

$$\kappa_m^{(k)}(\mathbf{r}_a) = \sum_{p \in Polars}\int_{V_m} \mathbf{G}^{ej}(\mathbf{j}_p) \cdot \mathbf{G}^{e}\left(\left(\frac{\partial\Phi^{(k)}}{\partial\mathbf{E}_p}, \frac{\partial\Phi^{(k)}}{\partial\mathbf{H}_p}\right)\delta_{\mathbf{r}_a}\right) dv(\mathbf{r}). \tag{41}$$

The reasoning to change the order of the Green's cofactors in eq. (41) compared to eq. (39) is dictated by generalization of the above formulae for the anisotropic case which we will discuss in Section 7.1. Note that from eq. (40) it immediately follows that

$$\frac{\partial\Phi^{(k)}}{\partial\sigma_m} = \kappa_m^{(k)}(\mathbf{r}_a). \tag{42}$$

5 DERIVATIVES OF THE MISFIT OF THE RESPONSE FUNCTION

Now we are equipped to calculate the differential of the misfit, defined by eq. (21), with respect to variation of σ. This differential can be written in the form (see Appendix D, eq. D3)

$$d\phi_d = 2\mathrm{Re}\left\{\sum_{\omega\in\Omega}\ \sum_{k \in Resps}\ \sum_{a \in Sites}\left(\Phi_a^{(k)}(\omega) - \Phi_a^{(k),exp}(\omega)\right)^* D_a^{(k)}(\omega)\ d\left(\Phi_a^{(k)}(\omega)\right)\right\}, \tag{43}$$

where upper asterisk stands for the complex conjugate. Substituting eq. (39) into the latter equation and rearranging the terms we obtain

$$d\phi_d = 2\mathrm{Re}\sum_{\omega\in\Omega}\sum_{p\in Polars}\left\langle \mathbf{G}^e\left(\mathbf{J}_p^E(\omega),\mathbf{J}_p^H(\omega)\right),\quad d\sigma\ \mathbf{G}^{ej}(\mathbf{j}_p(\omega))\right\rangle, \tag{44}$$

where

$$\mathbf{J}_p^E(\omega) = \sum_{a\in Sites}\sum_{k\in Resps}\left(\Phi_a^{(k)} - \Phi_a^{(k),exp}\right)^* D_a^{(k)}\frac{\partial\Phi_a^{(k)}}{\partial\mathbf{E}_p}\delta_{\mathbf{r}_a}, \tag{45}$$

$$\mathbf{J}_p^H(\omega) = \sum_{a\in Sites}\sum_{k\in Resps}\left(\Phi_a^{(k)} - \Phi_a^{(k),exp}\right)^* D_a^{(k)}\frac{\partial\Phi_a^{(k)}}{\partial\mathbf{H}_p}\delta_{\mathbf{r}_a}. \tag{46}$$

Again remembering the form of our model parametrization (see Section 2.3) along with eqs (44)–(46) we obtain

$$d\phi_d = 2\mathrm{Re}\sum_{m\in Model}\lambda_m\, d\sigma_m, \tag{47}$$

$$\lambda_m = \sum_{\omega\in\Omega}\sum_{p\in Polars}\int_{V_m}\mathbf{G}^{ej}(\mathbf{j}_p)\cdot\mathbf{G}^e\left(\mathbf{J}_p^E,\mathbf{J}_p^H\right)dv(\mathbf{r}). \tag{48}$$

From eq. (47) it follows that

$$\frac{\partial\phi_d}{\partial\mathrm{Re}\sigma_m} = 2\mathrm{Re}\ \lambda_m, \tag{49}$$

$$\frac{\partial\phi_d}{\partial\mathrm{Im}\sigma_m} = -2\mathrm{Im}\ \lambda_m. \tag{50}$$

Introducing the notation

$$\mathbf{E}_p^A = \mathbf{G}^e\left(\mathbf{J}_p^E,\mathbf{J}_p^H\right), \tag{51}$$

we rewrite eqs (49) and (50) as follows

$$\frac{\partial\phi_d}{\partial\mathrm{Re}\sigma_m} = 2\mathrm{Re}\sum_{\omega\in\Omega}\sum_{p\in Polars}\int_{V_m}\left(E_{xp}(\mathbf{r})E_{xp}^A(\mathbf{r}) + E_{yp}(\mathbf{r})E_{yp}^A(\mathbf{r}) + E_{zp}(\mathbf{r})E_{zp}^A(\mathbf{r})\right)dv(\mathbf{r}), \tag{52}$$

$$\frac{\partial\phi_d}{\partial\mathrm{Im}\sigma_m} = -2\mathrm{Im}\sum_{\omega\in\Omega}\sum_{p\in Polars}\int_{V_m}\left(E_{xp}(\mathbf{r})E_{xp}^A(\mathbf{r}) + E_{yp}(\mathbf{r})E_{yp}^A(\mathbf{r}) + E_{zp}(\mathbf{r})E_{zp}^A(\mathbf{r})\right)dv(\mathbf{r}). \tag{53}$$

These equations demonstrate the essence of the adjoint approach: in order to calculate gradient of the misfit one needs to perform only one (per frequency and polarization) additional forward modelling with the excitation provided by the adjoint source, which is determined via residuals of the response functions (see eqs 45 and 46).

6 EXAMPLES

In this section, we show how the general formalism works in practice.

6.1 Geomagnetic deep soundings (GDS)

Let H_r, H_ϑ and H_φ be the components of the magnetic field in the spherical (geographic) coordinate system. Let $\mathbf{e}_r$, $\mathbf{e}_\vartheta$ and $\mathbf{e}_\varphi$ be unit vectors of this coordinate system. Then *Z:H C*-responses (see eq. 18) can be written as

$$\widehat{C}_a(\omega) = \left(K\frac{H_r}{U}\right)\Bigg|_{\mathbf{r}_a}. \tag{54}$$

Here

$$K = -\tan\vartheta_d\frac{R_e}{2}, \tag{55}$$

and

$$U = \cos\alpha H_\vartheta - \sin\alpha H_\varphi, \tag{56}$$

where α is the angle between geographic and geomagnetic coordinate systems. For this example eqs (40) and (41) are reduced to

$$d\widehat{C}|_{\mathbf{r}_a} = \sum_{m\in Model}\kappa_m(\mathbf{r}_a)d\sigma_m, \tag{57}$$

$$\kappa_m(\mathbf{r}_a) = \int_{V_m} \mathbf{G}^{ej}(\mathbf{j}^{ext}) \cdot \mathbf{G}^{eh}\left(\frac{\partial \widehat{C}}{\partial \mathbf{H}} \delta_{\mathbf{r}_a}\right) dv(\mathbf{r}), \tag{58}$$

where $\mathbf{j}^{ext}(\mathbf{r})$, $\mathbf{r} \in \mathbb{R}^3$ is the large-scale magnetospheric source (ring current). According to eq. (27) we can write

$$\frac{\partial \widehat{C}}{\partial \mathbf{H}} = \frac{\partial \widehat{C}}{\partial H_r}\mathbf{e}_r + \frac{\partial \widehat{C}}{\partial H_\vartheta}\mathbf{e}_\vartheta + \frac{\partial \widehat{C}}{\partial H_\varphi}\mathbf{e}_\varphi, \tag{59}$$

and further, utilizing eqs (54) and (56), we have

$$\frac{\partial \widehat{C}}{\partial \mathbf{H}} = K\left[\frac{1}{U}\mathbf{e}_r - \frac{H_r}{U^2}\left(\cos\alpha\,\mathbf{e}_\vartheta - \sin\alpha\,\mathbf{e}_\varphi\right)\right]. \tag{60}$$

Substituting eq. (60) into eqs (57) and (58) we obtain

$$d\widehat{C}|_{\mathbf{r}_a} = \sum_{m \in Model} \kappa_m(\mathbf{r}_a)\, d\sigma_m, \tag{61}$$

$$\kappa_m(\mathbf{r}_a) = \int_{V_m} \mathbf{G}^{ej}(\mathbf{j}^{ext}) \cdot \mathbf{G}^{eh}\left(K\left(\frac{1}{U}\mathbf{e}_r - \frac{H_r}{U^2}\left(\cos\alpha\,\mathbf{e}_\vartheta - \sin\alpha\,\mathbf{e}_\varphi\right)\right)\delta_{\mathbf{r}_a}\right) dv(\mathbf{r}). \tag{62}$$

Ultimately eqs (47) and (48) in the scenario of GDS take the form

$$d\phi_d = 2\mathrm{Re} \sum_{m \in Model} \lambda_m^{eh}\, d\sigma_m, \tag{63}$$

$$\lambda_m^{eh} = \sum_{\omega \in \Omega} \int_{V_m} \mathbf{G}^{ej}(\mathbf{j}^{ext}(\omega)) \cdot \mathbf{G}^{eh}(\mathbf{J}^H(\omega)) dv(\mathbf{r}), \quad m \in Model, \tag{64}$$

$$\mathbf{J}^H = \sum_{a \in Sites} \left(\widehat{C}_a - \widehat{C}_a^{exp}\right)^* D_a\, K\left(\frac{1}{U}\mathbf{e}_r - \frac{H_r}{U^2}\left(\cos\alpha\,\mathbf{e}_\vartheta - \sin\alpha\,\mathbf{e}_\varphi\right)\right)\delta_{\mathbf{r}_a}. \tag{65}$$

It is seen from eqs (63) to (64) that the calculation of the misfit gradient in the GDS case requires one extra forward modelling per frequency component with the source $\mathbf{J}^H(\omega)$, described in (65).

6.2 Conventional MT

In the conventional MT method, the customary response functions are the elements of impedance tensor. Predicted elements of this tensor are calculated as follows

$$\mathbf{C} = \begin{pmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{pmatrix} = \mathbf{RQ},\ \mathbf{R} = \begin{pmatrix} E_{x1} & E_{x2} \\ E_{y1} & E_{y2} \end{pmatrix},\ \mathbf{Q} = \mathbf{S}^{-1},\ \mathbf{S} = \begin{pmatrix} H_{x1} & H_{x2} \\ H_{y1} & H_{y2} \end{pmatrix}. \tag{66}$$

Here $E_{x1},\ E_{y1},\ E_{x2},\ E_{y2}$ and $H_{x1},\ H_{y1},\ H_{x2},\ H_{y2}$ are, respectively, the horizontal components of electric and magnetic fields due to two primary plane waves of different polarization normally incident on the air-ground interface. In this case, eqs (40) and (41), with the use of the results obtained in Appendix C (see eq. C8), can be transformed as

$$dZ_{\xi\eta}|_{\mathbf{r}_a} = \sum_{m \in Model} \kappa_{\xi\eta m}^{a}\, d\sigma_m, \qquad \xi, \eta = x, y, \tag{67}$$

$$\kappa_{\xi\eta m}^{a} = \sum_{p=1}^{2} \int_{V_m} \mathbf{G}^{ej}(\mathbf{j}_p) \cdot \mathbf{G}^{e}\left(\mathbf{e}_{\xi\eta}^{pa}\delta_{\mathbf{r}_a}, \mathbf{h}_{\xi\eta}^{pa}\delta_{\mathbf{r}_a}\right) dv(\mathbf{r}), \tag{68}$$

where

$$\mathbf{h}_{\xi\eta}^{pa} = \left.\frac{\partial Z_{\xi\eta}}{\partial \mathbf{H}_p}\right|_{\mathbf{r}_a} = -Q_{p\eta}(Z_{\xi x}\mathbf{e}_x + Z_{\xi y}\mathbf{e}_y)|_{\mathbf{r}_a}, \tag{69}$$

$$\mathbf{e}_{\xi\eta}^{pa} = \left.\frac{\partial Z_{\xi\eta}}{\partial \mathbf{E}_p}\right|_{\mathbf{r}_a} = Q_{p\eta}\mathbf{e}_\xi|_{\mathbf{r}_a}, \qquad \xi, \eta = x, y; \quad p = 1, 2; \quad a \in Sites, \tag{70}$$

where $Z_{\xi\eta}$ and $Q_{p\eta}$ are calculated via eq. (66). Notation $\xi = x, y$ means that variable ξ runs through the alphabetical values, 'x' and 'y'. Hereafter the quotes are omitted for brevity. Finally for $d\phi_d$ we have

$$d\phi_d = 2\mathrm{Re} \sum_{m \in Model} \lambda_m\, d\sigma_m, \tag{71}$$

$$\lambda_m = \sum_{p=1}^{2} \sum_{\omega \in \Omega} \int_{V_m} \mathbf{G}^{ej}(\mathbf{j}_p(\omega)) \cdot \mathbf{G}^{e}\left(\mathbf{J}_p^E(\omega),\ \mathbf{J}_p^H(\omega)\right)\ dv(\mathbf{r}), \tag{72}$$

$$\mathbf{J}_p^E = \sum_{a \in Sites} \sum_{\substack{\xi = x,y \\ \eta = x,y}} \left(Z_{\xi\eta a} - Z_{\xi\eta a}^{exp} \right)^* D_{\xi\eta}^a \mathbf{e}_{\xi\eta}^{pa} \delta_{\mathbf{r}_a}, \tag{73}$$

$$\mathbf{J}_p^H = \sum_{a \in Sites} \sum_{\substack{\xi = x,y \\ \eta = x,y}} \left(Z_{\xi\eta a} - Z_{\xi\eta a}^{exp} \right)^* D_{\xi\eta}^a \mathbf{h}_{\xi\eta}^{pa} \delta_{\mathbf{r}_a}. \tag{74}$$

Thus to calculate the misfit gradient in the MT case one needs one extra forward modelling per frequency and per polarization with respective sources ($\mathbf{J}_p^E(\omega)$, $\mathbf{J}_p^H(\omega)$) described in (73) and (74).

6.3 Generalized MT

Recently Dmitriev & Berdichevsky (2002) considered a generalized model of impedance involving the inducing field containing a vertical magnetic component. In this case the MT tensor connects the horizontal components of the electric field with all components of the magnetic field

$$\begin{cases} E_x = Z_{xx} H_x + Z_{xy} H_y + Z_{xz} H_z, \\ E_y = Z_{yx} H_x + Z_{yy} H_y + Z_{yz} H_z. \end{cases} \tag{75}$$

Predicted elements of impedance tensor are calculated as follows

$$\mathbf{C} = \begin{pmatrix} Z_{xx} & Z_{xy} & Z_{xz} \\ Z_{yx} & Z_{yy} & Z_{yz} \end{pmatrix} = \mathbf{RQ}, \quad \mathbf{R} = \begin{pmatrix} E_{x1} & E_{x2} & E_{x3} \\ E_{y1} & E_{y2} & E_{y3} \end{pmatrix}, \quad \mathbf{Q} = \mathbf{S}^{-1}, \quad \mathbf{S} = \begin{pmatrix} H_{x1} & H_{x2} & H_{x3} \\ H_{y1} & H_{y2} & H_{y3} \\ H_{z1} & H_{z2} & H_{z3} \end{pmatrix}. \tag{76}$$

Here E_{x1}, E_{y1}, E_{x2}, E_{y2} and H_{x1}, H_{y1}, H_{z1}, H_{x2}, H_{y2}, H_{z2} are, respectively, the horizontal components of the electric field and all components of the magnetic fields due to two primary plane waves of different polarization normally incident on the air-ground interface. In addition E_{x3}, E_{y3} and H_{x3}, H_{y3}, H_{z3} are the fields due to primary EM field whose horizontal components linearly vary along the surface and whose vertical magnetic component is not zero (for further details see Dmitriev & Berdichevsky 2002). Note that introducing this model one can considerably extend the capabilities of MT studies, particularly in regions that are close to the sources of the observed field, for example, in the auroral zones. For the generalized MT method eqs (40) and (41) are transformed as

$$dZ_{\xi\eta}|_{\mathbf{r}_a} = \sum_{m \in Model} \kappa_{\xi\eta m}^a d\sigma_m, \qquad \xi = x, y; \quad \eta = x, y, z, \tag{77}$$

$$\kappa_{\xi\eta m}^a = \sum_{p=1}^{3} \int_{V_m} \mathbf{G}^{ej}(\mathbf{j}_p) \cdot \mathbf{G}^e \left(\mathbf{e}_{\xi\eta p}^a \delta_{\mathbf{r}_a}, \mathbf{h}_{\xi\eta p}^a \delta_{\mathbf{r}_a} \right) dv(\mathbf{r}), \tag{78}$$

where for $\mathbf{h}_{\xi\eta p}^a$ and $\mathbf{e}_{\xi\eta p}^a$ we have

$$\mathbf{h}_{\xi\eta p}^a = \left. \frac{\partial Z_{\xi\eta}}{\partial \mathbf{H}_p} \right|_{\mathbf{r}_a} = -Q_{p\eta} \left(Z_{\xi x} \mathbf{e}_x + Z_{\xi y} \mathbf{e}_y + Z_{\xi z} \mathbf{e}_z \right) \Big|_{\mathbf{r}_a}, \tag{79}$$

$$\mathbf{e}_{\xi\eta p}^a = \left. \frac{\partial Z_{\xi\eta}}{\partial \mathbf{E}_p} \right|_{\mathbf{r}_a} = Q_{p\eta} \mathbf{e}_\xi \Big|_{\mathbf{r}_a}, \quad \xi = x, y; \quad \eta = x, y, z; \quad p = 1, 2, 3; \quad a \in Sites. \tag{80}$$

Finally for $d\phi_d$ we obtain

$$d\phi_d = 2\mathrm{Re} \sum_{m \in Model} \lambda_m \, d\sigma_m, \tag{81}$$

$$\lambda_m = \sum_{\omega \in \Omega} \sum_{p=1}^{3} \int_{V_m} \mathbf{G}^{ej}(\mathbf{j}_p(\omega)) \cdot \mathbf{G}^e \left(\mathbf{J}_p^E(\omega), \ \mathbf{J}_p^H(\omega) \right) \, dv(\mathbf{r}), \tag{82}$$

$$\mathbf{J}_p^E = \sum_{a \in Sites} \sum_{\substack{\xi = x,y \\ \eta = x,y,z}} \left(Z_{\xi\eta a} - Z_{\xi\eta a}^{exp} \right)^* D_{\xi\eta a} \mathbf{e}_{\xi\eta p}^a \delta_{\mathbf{r}_a}, \tag{83}$$

$$\mathbf{J}_p^H = \sum_{a \in Sites} \sum_{\substack{\xi = x,y \\ \eta = x,y,z}} \left(Z_{\xi\eta a} - Z_{\xi\eta a}^{exp} \right)^* D_{\xi\eta a} \mathbf{h}_{\xi\eta p}^a \delta_{\mathbf{r}_a}. \tag{84}$$

Thus to calculate the misfit gradient in generalized MT case one needs one extra forward modelling per frequency and per polarization with respective sources ($\mathbf{J}_p^E(\omega)$, $\mathbf{J}_p^H(\omega)$) described in (83) and (84).

6.4 MV tippers

The MV tipper connects the vertical magnetic component with horizontal magnetic components

$$H_z = W_x H_x + W_y H_y. \tag{85}$$

Predicted W_η are calculated as follows:

$$\mathbf{C} = (W_x, W_y) = \mathbf{RQ}, \qquad \mathbf{R} = (H_{z1}, H_{z2}), \qquad \mathbf{Q} = \mathbf{S}^{-1}, \qquad \mathbf{S} = \begin{pmatrix} H_{x1} & H_{x2} \\ H_{y1} & H_{y2} \end{pmatrix}. \tag{86}$$

For this case eqs (40) and (41) are transformed as

$$dW_\eta|_{\mathbf{r}_a} = \sum_{m\in Model} \kappa^a_{\eta m} d\sigma_m, \qquad \eta = x, y, \tag{87}$$

$$\kappa^a_{\eta m} = \sum_{p=1}^{2} \int_{V_m} \mathbf{G}^{ej}(\mathbf{j}_p) \cdot \mathbf{G}^{eh}\left(\mathbf{h}^a_{\eta p}\delta_{\mathbf{r}_a}\right) dv(\mathbf{r}), \qquad a \in Sites, \tag{88}$$

where $\mathbf{h}^a_{\eta p}$ is defined as

$$\mathbf{h}^a_{\eta p} = \left.\frac{\partial W_\eta}{\partial \mathbf{H}_p}\right|_{\mathbf{r}_a} = -Q_{p\eta}(W_x\mathbf{e}_x + W_y\mathbf{e}_y - \mathbf{e}_z)\Big|_{\mathbf{r}_a}, \qquad \eta = x, y; \quad p = 1, 2; \quad a \in Sites. \tag{89}$$

Finally for $d\phi_d$ we have

$$d\phi_d = 2\mathrm{Re} \sum_{m\in Model} \lambda_m \, d\sigma_m, \tag{90}$$

$$\lambda_m = \sum_{\omega\in\Omega} \sum_{p=1}^{2} \int_{V_m} \mathbf{G}^{ej}(\mathbf{j}_p(\omega)) \cdot \mathbf{G}^{eh}\left(\mathbf{J}^W_p(\omega)\right) dv(\mathbf{r}), \qquad m \in Model, \tag{91}$$

$$\mathbf{J}^W_p = \sum_{a\in Sites} \sum_{\eta=x,y} \left(W_{\eta a} - W^{exp}_{\eta a}\right)^* D^a_\eta \mathbf{h}^a_{\eta p}\delta_{\mathbf{r}_a}, \qquad p = 1, 2. \tag{92}$$

Thus to calculate the misfit gradient in the MV tipper case one needs one extra forward modelling per frequency and per polarization with respective source $\mathbf{J}^W_p(\omega)$ described in (92).

6.5 Horizontal magnetic tensor

So far we discussed single-site response functions that connect components of the EM field at the same site. In this section and Section 6.6 we consider another form of response functions, multisite responses that connect MT components at different sites. In particular in this section we deal with horizontal magnetic tensor that connects horizontal magnetic components at some pair of the observation sites, $\mathbf{r}_a$ and $\mathbf{r}_b$

$$\begin{cases} H_x(\mathbf{r}_b) = M^{ab}_{xx} H_x(\mathbf{r}_a) + M^{ab}_{xy} H_y(\mathbf{r}_a), \\ H_y(\mathbf{r}_b) = M^{ab}_{yx} H_x(\mathbf{r}_a) + M^{ab}_{yy} H_y(\mathbf{r}_a). \end{cases} \tag{93}$$

Predicted elements of horizontal magnetic tensor are calculated as follows:

$$\mathbf{C} = \begin{pmatrix} M^{ab}_{xx} & M^{ab}_{xy} \\ M^{ab}_{yx} & M^{ab}_{yy} \end{pmatrix} = \mathbf{RQ}, \qquad \mathbf{R} = \begin{pmatrix} H_{x1}(\mathbf{r}_b) & H_{x2}(\mathbf{r}_b) \\ H_{y1}(\mathbf{r}_b) & H_{y2}(\mathbf{r}_b) \end{pmatrix}, \qquad \mathbf{Q} = \mathbf{S}^{-1}, \qquad \mathbf{S} = \begin{pmatrix} H_{x1}(\mathbf{r}_a) & H_{x2}(\mathbf{r}_a) \\ H_{y1}(\mathbf{r}_a) & H_{y2}(\mathbf{r}_a) \end{pmatrix}. \tag{94}$$

In Appendix E, we extend the formalism (16)–(53) for the case of multisite responses. Letting $n = (\xi, \eta, a, b)$ and following (E6) we deduce that the differential $dM_{\xi\eta}(\mathbf{r}_a, \mathbf{r}_b)$ can be written as

$$dM^{ab}_{\xi\eta} = \sum_{p=1,2} \left\langle \mathbf{G}^{eh}\left(\frac{\partial M^{ab}_{\xi\eta}}{\partial \mathbf{H}_p(\mathbf{r}_a)}\delta_{\mathbf{r}_a} + \frac{\partial M^{ab}_{\xi\eta}}{\partial \mathbf{H}_p(\mathbf{r}_b)}\delta_{\mathbf{r}_b}\right), \; d\sigma\, \mathbf{G}^{ej}(\mathbf{j}_p)\right\rangle, \qquad \forall a, b \in Sites; \quad \xi, \eta = x, y. \tag{95}$$

Further, using eqs (C7) and (94), we have

$$d\mathbf{C} = (d\mathbf{R})\,\mathbf{Q} - \mathbf{C}\,(d\mathbf{S})\,\mathbf{Q}, \tag{96}$$

where

$$dM^{ab}_{\xi\eta} = \sum_{p=1,2} (dH_{\xi p}(\mathbf{r}_b))Q^a_{p\eta} - \sum_{\substack{p=1,2\\ \beta=x,y}} M^{ab}_{\xi\eta}(dH_{\beta p}(\mathbf{r}_a))Q^a_{p\eta}, \qquad \forall a, b \in Sites; \quad \xi, \eta = x, y. \tag{97}$$

Finally we can write the coefficients of the coupling matrix sensitivity as

$$\frac{\partial M^{ab}_{\xi\eta}}{\partial \mathbf{H}_p(\mathbf{r}_a)} = -Q^a_{p\eta}\left(M^{ab}_{\xi x}\mathbf{e}_x + M^{ab}_{\xi y}\mathbf{e}_y\right), \quad \frac{\partial M^{ab}_{\xi\eta}}{\partial \mathbf{H}_p(\mathbf{r}_b)} = Q^a_{p\eta}\mathbf{e}_\xi, \qquad \forall a, b \in Sites; \qquad p = 1, 2; \qquad \xi, \eta = x, y. \tag{98}$$

For this case eqs (40) and (41) are transformed as follows

$$dM^{ab}_{\xi\eta} = \sum_{m\in Model} \kappa^{ab}_{\xi\eta m} d\sigma_m, \qquad \xi,\eta = x, y. \tag{99}$$

$$\kappa^{ab}_{\xi\eta m} = \sum_{p=1}^{2} \int_{V_m} \mathbf{G}^{ej}(\mathbf{j}_p) \cdot \mathbf{G}^{eh}\left(\mathbf{h}^{ab}_{\xi\eta p1}\delta_{\mathbf{r}_a} + \mathbf{h}^{ab}_{\xi\eta p2}\delta_{\mathbf{r}_b}\right) dv(\mathbf{r}), \qquad m \in Model, \tag{100}$$

where $\mathbf{h}^{ab}_{\xi\eta p1}$ and $\mathbf{h}^{ab}_{\xi\eta p2}$ are defined as

$$\mathbf{h}^{ab}_{\xi\eta p1} = \frac{\partial M^{ab}_{\xi\eta}}{\partial \mathbf{H}_p(\mathbf{r}_a)} = -Q^a_{p\eta}\left(M^{ab}_{\xi x}\mathbf{e}_x + M^{ab}_{\xi y}\mathbf{e}_y\right), \tag{101}$$

$$\mathbf{h}^{ab}_{\xi\eta p2} = \frac{\partial M^{ab}_{\xi\eta}}{\partial \mathbf{H}_p(\mathbf{r}_b)} = Q^a_{p\eta}\mathbf{e}_\xi, \qquad \xi,\eta = x, y; \quad p = 1, 2; \quad a = a_i,\ b = b_i \in Sites, \quad i = 1, \ldots, N. \tag{102}$$

Finally for $d\phi_d$ we have

$$d\phi_d = 2\mathrm{Re} \sum_{m\in Model} \lambda_m\, d\sigma_m, \tag{103}$$

$$\lambda_m = \sum_{\omega\in\Omega}\sum_{p=1}^{2} \int_{V_m} \mathbf{G}^{ej}(\mathbf{j}_p(\omega)) \cdot\ \mathbf{G}^{eh}\left(\mathbf{J}^M_p(\omega)\right) dv(\mathbf{r}), \qquad m \in Model, \tag{104}$$

$$\mathbf{J}^M_p = \sum_{i=1}^{N} \sum_{\substack{\xi=x,y\\ \eta=x,y}} \left(M^{a_i b_i}_{\xi\eta} - M^{a_i b_i, exp}_{\xi\eta}\right)^* D^{a_i b_i}_{\xi\eta}\ \left(\mathbf{h}^{a_i b_i}_{\xi\eta 1}\delta_{\mathbf{r}_{a_i}} + \mathbf{h}^{a_i b_i}_{\xi\eta 2}\delta_{\mathbf{r}_{b_i}}\right), \qquad p = 1, 2, \tag{105}$$

where $(\mathbf{r}_{a_1}, \mathbf{r}_{b_1}), \ldots, (\mathbf{r}_{a_N}, \mathbf{r}_{b_N})$ are the pairs of observation sites where horizontal magnetic tensor is evaluated. Let us note that $\mathbf{r}_{a_i}$ and $\mathbf{r}_{b_i}$ do not necessarily differ from $\mathbf{r}_{a_j}$ and $\mathbf{r}_{b_j}$, so that any kind of full-matrix observation scheme is acceptable. For example $\mathbf{r}_a$ might be some fixed (reference) point.

Thus to calculate the misfit gradient in case of the horizontal magnetic tensor one needs one extra forward modelling per frequency and per polarization with the source $\mathbf{J}^M_p(\omega)$ described in (105).

6.6 Horizontal electric tensor

In the section we consider the horizontal electric tensor that connects the horizontal electric components at two sites

$$\begin{cases} E_x(\mathbf{r}_b) = T^{ab}_{xx} E_x(\mathbf{r}_a) + T^{ab}_{xy} E_y(\mathbf{r}_a), \\ E_y(\mathbf{r}_b) = T^{ab}_{yx} E_x(\mathbf{r}_a) + T^{ab}_{yy} E_y(\mathbf{r}_a). \end{cases} \tag{106}$$

Predicted elements of the horizontal electric tensor are calculated as follows

$$\mathbf{C} = \begin{pmatrix} T^{ab}_{xx} & T^{ab}_{xy} \\ T^{ab}_{yx} & T^{ab}_{yy} \end{pmatrix} = \mathbf{RQ}, \qquad \mathbf{R} = \begin{pmatrix} E_{x1}(\mathbf{r}_b) & E_{x2}(\mathbf{r}_b) \\ E_{y1}(\mathbf{r}_b) & E_{y2}(\mathbf{r}_b) \end{pmatrix}, \qquad \mathbf{Q} = \mathbf{S}^{-1}, \qquad \mathbf{S} = \begin{pmatrix} E_{x1}(\mathbf{r}_a) & E_{x2}(\mathbf{r}_a) \\ E_{y1}(\mathbf{r}_a) & E_{y2}(\mathbf{r}_a) \end{pmatrix}. \tag{107}$$

Since the derivation of differentials for this case is very similar to the derivation discussed in previous section we present here only the resulting formulae

$$dT^{ab}_{\xi\eta} = \sum_{m\in Model} \kappa^{ab}_{\xi\eta m} d\sigma_m, \qquad \xi,\eta = x, y; \ \ a = a_i,\ b = b_i \in Sites, \tag{108}$$

$$\kappa^{ab}_{\xi\eta m} = \sum_{p=1}^{2} \int_{V_m} \mathbf{G}^{ej}(\mathbf{j}_p) \cdot \mathbf{G}^{ej}\left(\mathbf{e}^{ab}_{\xi\eta p1}\delta_{\mathbf{r}_a} + \mathbf{e}^{ab}_{\xi\eta p2}\delta_{\mathbf{r}_b}\right) dv(\mathbf{r}), \tag{109}$$

where $\mathbf{e}^{ab}_{\xi\eta p1}$ and $\mathbf{e}^{ab}_{\xi\eta p2}$ are

$$\mathbf{e}^{ab}_{\xi\eta p1} = \frac{\partial T^{ab}_{\xi\eta}}{\partial \mathbf{E}_p(\mathbf{r}_a)} = -Q^a_{p\eta}\left(T^{ab}_{\xi x}\mathbf{e}_x + T^{ab}_{\xi y}\mathbf{e}_y\right), \tag{110}$$

$$\mathbf{e}^{ab}_{\xi\eta p2} = \frac{\partial T^{ab}_{\xi\eta}}{\partial \mathbf{E}_p(\mathbf{r}_b)} = Q^a_{p\eta}\mathbf{e}_\xi, \qquad \xi,\eta = x, y; \quad p = 1, 2; \quad a = a_i,\ b = b_i \in Sites. \tag{111}$$

Finally for $d\phi_d$ we have

$$d\phi_d = 2\mathrm{Re} \sum_{m\in Model} \lambda_m\, d\sigma_m, \tag{112}$$

$$\lambda_m = \sum_{\omega \in \Omega} \sum_{p=1}^{2} \int_{V_m} \mathbf{G}^{ej}(\mathbf{j}_p(\omega)) \cdot \mathbf{G}^{ej}\left(\mathbf{J}_p^E(\omega)\right) dv(\mathbf{r}), \qquad m \in Model, \tag{113}$$

$$\mathbf{J}_p^E = \sum_{i=1}^{N} \sum_{\substack{\xi=x,y \\ \eta=x,y}} \left(T_{\xi\eta}^{a_i b_i} - T_{\xi\eta}^{a_i b_i, exp}\right)^* D_{\xi\eta}^{a_i b_i} \left(\mathbf{e}_{\xi\eta p1}^{a_i b_i} \delta_{\mathbf{r}_{a_i}} + \mathbf{e}_{\xi\eta p2}^{a_i b_i} \delta_{\mathbf{r}_{b_i}}\right), \qquad p = 1, 2. \tag{114}$$

Thus to calculate the misfit gradient in case of horizontal electric tensor one needs one extra forward modelling per frequency and per polarization with the source $\mathbf{J}_p^E(\omega)$ described in (114).

6.7 Horizontal gradient sounding

In HGS the conventional response function is local C-response, which is determined as (*cf*. Schmucker 1970; Olsen 1998)

$$\widehat{C}_a(\omega) \equiv \widehat{C}(\mathbf{r}, \omega)|_{\mathbf{r}_a} = \left(-\frac{H_r}{\mathrm{div}_\tau\ \mathbf{H}}\right)\Big|_{\mathbf{r}_a}, \tag{115}$$

where $\mathrm{div}_\tau\ \mathbf{H}$ stands for the angular part of divergence. This case requires special attention since the response function depends now not only on the magnetic field but also on spatial derivatives of the magnetic field. In Appendix F we extend the formalism of (16)–(53) to the case of responses that contain spatial derivatives. Thus, by denoting

$$\widehat{C} = -H_r/u \tag{116}$$

$$u = \mathrm{div}_\tau\ \mathbf{H}, \tag{117}$$

$$L_a(\mathbf{H}) = \mathrm{div}_\tau\ \mathbf{H}\big|_{\mathbf{r}_a} = \frac{1}{r \sin\theta} \left(\frac{\partial (\sin\theta\ H_\theta)}{\partial\theta} + \frac{\partial H_\varphi}{\partial\varphi}\right)\Big|_{\mathbf{r}_a}, \qquad \forall a \in Sites, \tag{118}$$

and following eqs (F3), (F13) and (5) we deduce that differentials $d\widehat{C}(\mathbf{r}_a)$ can be written as

$$d\widehat{C}(\mathbf{r}_a) = \left\langle \mathbf{G}^{eh}\left(\frac{\partial \widehat{C}}{\partial H_r} \mathbf{e}_r \delta_{\mathbf{r}_a} - \frac{\partial \widehat{C}}{\partial u} \nabla_\tau \delta_{\mathbf{r}_a}\right),\ d\sigma\ \mathbf{G}^{ej}\left(\mathbf{j}^{ext}\right)\right\rangle, \qquad \forall a \in Sites, \tag{119}$$

where

$$\frac{\partial \widehat{C}}{\partial H_r} = -\frac{1}{u}, \qquad \frac{\partial \widehat{C}}{\partial u} = \frac{H_r}{u^2}. \tag{120}$$

Following eqs (F4) and (F5) we further get for $d\widehat{C}$

$$d\widehat{C}|_{\mathbf{r}_a} = \sum_{m \in Model} \kappa_m^a d\sigma_m, \tag{121}$$

$$\kappa_m^a = -\int_{V_m} \mathbf{G}^{ej}(\mathbf{j}^{ext}) \cdot \mathbf{G}^{eh}\left(\frac{1}{u}\mathbf{e}_r \delta_{\mathbf{r}_a} + \frac{H_r}{u^2} \nabla_\tau \delta_{\mathbf{r}_a}\right) dv(\mathbf{r}), \tag{122}$$

and for $d\phi_d$ (from eqs F9 to F11)

$$d\phi_d = 2\mathrm{Re} \sum_{m \in Model} \lambda_m^{eh}\ d\sigma_m, \tag{123}$$

$$\lambda_m^{eh} = \sum_{\omega \in \Omega} \int_{V_m} \mathbf{G}^{ej}(\mathbf{j}^{ext}(\omega)) \cdot \mathbf{G}^{eh}(\mathbf{J}^H(\omega)) dv(\mathbf{r}), \tag{124}$$

$$\mathbf{J}^H = -\sum_{a \in Sites} \left(\widehat{C}_a - \widehat{C}_a^{exp}\right)^* D_a \left(\frac{1}{u}\mathbf{e}_r \delta_{\mathbf{r}_a} + \frac{H_r}{u^2} \nabla_\tau \delta_{\mathbf{r}_a}\right). \tag{125}$$

Thus to calculate the misfit gradient in the HGS case one needs one extra forward modelling per frequency with the source $\mathbf{J}^H(\omega)$ described in (125).

6.8 Combination of GDS and HGS

The responses described in Sections 6.1 and 6.7 have been designed to deal with 1-D models of the Earth. Recently, Schmucker (2003) promoted a generalization of this approach that removes the constraint about one-dimensionality of the Earth. This generalization introduces a new ratio, which locally relates the radial magnetic component with both spatial derivatives of the horizontal components and the horizontal components themselves. In a simplified form (*cf*. Semenov *et al.* 2007) this ratio looks like

$$H_r = C_u u + C_\vartheta H_\vartheta + C_\varphi H_\varphi, \tag{126}$$

where u is determined by eq. (117). In accordance with Appendix B predictions of $C_u,\ C_\vartheta,\ C_\varphi$ are calculated as follows

$$\mathbf{C} = \left(C_u,\ C_\vartheta,\ C_\varphi\right) = \mathbf{RQ}, \qquad \mathbf{R} = (H_{r1},\ H_{r2},\ H_{r3}), \qquad \mathbf{Q} = \mathbf{S}^{-1}, \qquad \mathbf{S} = \begin{pmatrix} u_1 & u_2 & u_3 \\ H_{\vartheta 1} & H_{\vartheta 2} & H_{\vartheta 3} \\ H_{\varphi 1} & H_{\varphi 2} & H_{\varphi 3} \end{pmatrix}. \tag{127}$$

Here $u_p,\ H_{rp},\ H_{\vartheta p},\ H_{\varphi p}$ are, respectively, the angular part of divergence, and the components of magnetic field due to pth source polarization, $\mathbf{j}_p$. In this case, eqs (C8) and (F4) and (F5) can be transformed as

$$dC_\beta|_{\mathbf{r}_a} = \sum_{m\in Model} \kappa^a_{\beta m} d\sigma_m, \quad \beta = u, \vartheta, \varphi, \tag{128}$$

$$\kappa^a_{\beta m} = \sum_{p=1}^{3} \int_{V_m} \mathbf{G}^{ej}(\mathbf{j}_p)\cdot \mathbf{G}^{eh}\left(\mathbf{q}^{pa}_\beta \delta_{\mathbf{r}_a} + v^{pa}_\beta \nabla_\tau \delta_{\mathbf{r}_a}\right) dv(\mathbf{r}), \qquad m \in Model. \tag{129}$$

where

$$\mathbf{q}^{pa}_\beta = \left.\frac{\partial C_\beta}{\partial \mathbf{H}_p}\right|_{\mathbf{r}_a} = \left.\frac{\partial C_\beta}{\partial H_{rp}}\mathbf{e}_r + \frac{\partial C_\beta}{\partial H_{\vartheta p}}\mathbf{e}_\vartheta + \frac{\partial C_\beta}{\partial H_{\varphi p}}\mathbf{e}_\varphi\right|_{\mathbf{r}_a} = Q_{p\beta}(\mathbf{e}_r - C_\vartheta \mathbf{e}_\vartheta - C_\varphi \mathbf{e}_\varphi)|_{\mathbf{r}_a}, \tag{130}$$

$$v^{pa}_\beta = \left.\frac{\partial C_\beta}{\partial u_p}\right|_{\mathbf{r}_a} = -C_u Q_{p\beta}\big|_{\mathbf{r}_a}, \qquad p = 1, 2; \quad \beta = u, \vartheta, \varphi; \quad a \in Sites. \tag{131}$$

Finally, in this case for $d\phi_d$ we have (from eqs F9–F11)

$$d\phi_d = 2\mathrm{Re} \sum_{m\in Model} \lambda_m\, d\sigma_m, \tag{132}$$

$$\lambda_m = \sum_{\omega\in\Omega}\sum_{p=1}^{3} \int_{V_m} \mathbf{G}^{ej}(\mathbf{j}_p(\omega))\cdot \mathbf{G}^{eh}\left(\mathbf{J}^H_p(\omega)\right) dv(\mathbf{r}), \qquad m \in Model, \tag{133}$$

$$\mathbf{J}^H_p = \sum_{a\in Sites}\ \sum_{\beta=u,\vartheta,\varphi} \left(C_{\beta a} - C^{exp}_{\beta a}\right)^* D_{\beta a}\left(\mathbf{q}^{pa}_\beta \delta_{\mathbf{r}_a} + v^{pa}_\beta \nabla_\tau \delta_{\mathbf{r}_a}\right), \qquad p = 1, 2, 3. \tag{134}$$

Thus to calculate the misfit gradient in the 'GDS+HGS' case one needs one extra forward modelling per frequency and per polarization with the source $\mathbf{J}^H_p(\omega)$ described in (134).

7 DISCUSSION

7.1 Generalization for an anisotropic case

So far we assumed that conductivity is a scalar-valued function (the isotropic case). However, it is important to stress that the derivation of eqs (29), (32) and the resulting eqs (39) and (44)–(45) is valid for more general anisotropic case when the conductivity is a matrix-valued function, that is,

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}. \tag{135}$$

Note, that in this section, for simplicity, but without loss of generality, we consider Cartesian coordinate system. Modifications of eqs (40) and (41) (and their extensions E7–E14 and F4–F11) in anisotropic case are three-fold: (1) the coefficients κ and λ become matrix-valued; (2)

the matrix trace, Tr(·), should be applied to the first equation of (40); and (3) the scalar product '·' of Green's cofactors in equations of (41) should be substituted by tensor product '⊗'. These modifications give

$$d\Phi^{(k)}\big|_{\mathbf{r}_a} = \mathrm{Tr} \sum_{m\in Model} \kappa_m^{(k)}(\mathbf{r}_a) d\sigma_m, \tag{136}$$

$$\kappa_m^{(k)}(\mathbf{r}_a) = \sum_{p\in Polars} \int_{V_m} \mathbf{G}^{ej}(\mathbf{j}_p) \otimes \mathbf{G}^{e}\left(\left(\frac{\partial\Phi^{(k)}}{\partial\mathbf{E}_p}, \frac{\partial\Phi^{(k)}}{\partial\mathbf{H}_p}\right)\delta_{\mathbf{r}_a}\right) dv(\mathbf{r}), \tag{137}$$

where the tensor product of two vector columns is a matrix defined as follows

$$\mathbf{u}\otimes\mathbf{v} = \mathbf{u}\mathbf{v}^T = \begin{pmatrix} u_x v_x & u_x v_y & u_x v_z \\ u_y v_x & u_y v_y & u_y v_z \\ u_z v_x & u_z v_y & u_z v_z \end{pmatrix}, \tag{138}$$

and the term $\kappa_m^{(k)}(\mathbf{r}_a)d\sigma_m$ in eq. (136) stands for the matrix multiplication of 3 × 3 matrices $\kappa_m^{(k)}(\mathbf{r}_a)$ and $d\sigma_m$ (*cf*. eq. 135). From eq. (136) it follows that the sensitivity coefficients of $\Phi^{(k)}$ with respect to σ_{ijm} are

$$\frac{\partial\Phi_a^{(k)}}{\partial\sigma_{xxm}} = \kappa_{xxm}^{(k),a}, \quad \frac{\partial\Phi_a^{(k)}}{\partial\sigma_{xym}} = \kappa_{yxm}^{(k),a}, \quad \ldots, \quad \frac{\partial\Phi_a^{(k)}}{\partial\sigma_{zzm}} = \kappa_{zzm}^{(k),a}, \qquad m\in Model, \quad a\in Sites, \quad k\in Resps. \tag{139}$$

Corresponding modifications must be applied to equations in (47) yielding

$$d\phi_d = 2\mathrm{Re}\,\mathrm{Tr}\sum_{m\in Model} \lambda_m\, d\sigma_m, \tag{140}$$

$$\lambda_m = \sum_{\omega\in\Omega}\sum_{p\in Polars}\int_{V_m} \mathbf{G}^{ej}\left(\mathbf{j}_p(\omega)\right)\otimes\mathbf{G}^{e}\left(\mathbf{J}_p^E(\omega), \mathbf{J}_p^H(\omega)\right) dv(\mathbf{r}), \qquad m\in Model, \tag{141}$$

$$\mathbf{J}_p^E = \sum_{a\in Sites}\sum_{k\in Resps}\left(\Phi_a^{(k)} - \Phi_a^{(k),\ exp}\right)^* D_a^{(k)} \frac{\partial\Phi_a^{(k)}}{\partial\mathbf{E}_p}\delta_{\mathbf{r}_a}, \tag{142}$$

$$\mathbf{J}_p^H = \sum_{a\in Sites}\sum_{k\in Resps}\left(\Phi_a^{(k)} - \Phi_a^{(k),\ exp}\right)^* D_a^{(k)} \frac{\partial\Phi_a^{(k)}}{\partial\mathbf{H}_p}\delta_{\mathbf{r}_a}, \qquad p\in Polars. \tag{143}$$

The same modifications should be applied to the all formulae for the different scenarios described in Section 6.

From eq. (140) it follows that the sensitivity coefficients of ϕ_d with respect to $\mathrm{Re}\,\sigma_{ijm}$ and $\mathrm{Im}\,\sigma_{ijm}$ are

$$\left.\begin{array}{r} \dfrac{\partial\phi_d}{\partial\mathrm{Re}\,\sigma_{xxm}} = 2\mathrm{Re}\ \lambda_{xxm}, \\ \dfrac{\partial\phi_d}{\partial\mathrm{Im}\,\sigma_{xxm}} = -2\mathrm{Im}\ \lambda_{xxm}, \\ \dfrac{\partial\phi_d}{\partial\mathrm{Re}\,\sigma_{xym}} = 2\mathrm{Re}\ \lambda_{yxm}, \\ \dfrac{\partial\phi_d}{\partial\mathrm{Im}\,\sigma_{xym}} = -2\mathrm{Im}\ \lambda_{yxm}, \\ \ldots \\ \dfrac{\partial\phi_d}{\partial\mathrm{Re}\,\sigma_{zzm}} = 2\mathrm{Re}\ \lambda_{zzm}, \\ \dfrac{\partial\phi_d}{\partial\mathrm{Im}\,\sigma_{zzm}} = -2\mathrm{Im}\ \lambda_{zzm}. \end{array}\right\} m\in Model. \tag{144}$$

Remembering notation of $\mathbf{E}_p^A$ (eq. 51), we can re-interpret equations in (144) as

$$\left.\begin{aligned}
\frac{\partial\phi_d}{\partial \mathrm{Re}\,\sigma_{xxm}} &= 2\mathrm{Re}\sum_{\omega\in\Omega}\sum_{p\in Polars}\int_{V_m} E_{xp}(\mathbf{r})E_{xp}^A(\mathbf{r})dv(\mathbf{r}),\\
\frac{\partial\phi_d}{\partial \mathrm{Im}\,\sigma_{xxm}} &= -2\mathrm{Im}\sum_{\omega\in\Omega}\sum_{p\in Polars}\int_{V_m} E_{xp}(\mathbf{r})E_{xp}^A(\mathbf{r})dv(\mathbf{r}),\\
\frac{\partial\phi_d}{\partial \mathrm{Re}\,\sigma_{xym}} &= 2\mathrm{Re}\sum_{\omega\in\Omega}\sum_{p\in Polars}\int_{V_m} E_{yp}(\mathbf{r})E_{xp}^A(\mathbf{r})dv(\mathbf{r}),\\
\frac{\partial\phi_d}{\partial \mathrm{Im}\,\sigma_{xym}} &= -2\mathrm{Im}\sum_{\omega\in\Omega}\sum_{p\in Polars}\int_{V_m} E_{yp}(\mathbf{r})E_{xp}^A(\mathbf{r})dv(\mathbf{r}),\\
&\dots\\
\frac{\partial\phi_d}{\partial \mathrm{Re}\,\sigma_{zzm}} &= 2\mathrm{Re}\sum_{\omega\in\Omega}\sum_{p\in Polars}\int_{V_m} E_{zp}(\mathbf{r})E_{zp}^A(\mathbf{r})dv(\mathbf{r}),\\
\frac{\partial\phi_d}{\partial \mathrm{Im}\,\sigma_{zzm}} &= -2\mathrm{Im}\sum_{\omega\in\Omega}\sum_{p\in Polars}\int_{V_m} E_{zp}(\mathbf{r})E_{zp}^A(\mathbf{r})dv(\mathbf{r}).
\end{aligned}\right\}\ m \in Model. \tag{145}$$

7.2 Alternative model parametrizations

So far we exploited model parametrization described in Section 2.3 (see eq. 12). However the common practice is to define the model parameters in a form

$$\mathbf{m} = \left(\ln\sigma_1,\ \ln\sigma_2, \dots,\ \ln\sigma_{N_M}\right), \tag{146}$$

where ln denotes the natural logarithm. In view of the relation $d(\ln\sigma_m) = \frac{d\sigma_m}{\sigma_m}$, the required modifications of the resulting equations are straightforward.

Another option to parametrize the model is to decompose the conductivity distribution in a series of some a priori given spatial forms (e.g. Fourier series)

$$\sigma(\mathbf{r}) = \sum_{q=1}^{N_Q} s_q\Psi_q(\mathbf{r}). \tag{147}$$

In this case model parameters look as follows

$$\mathbf{m} = \left(s_1,\ s_2, \dots,\ s_{N_Q}\right). \tag{148}$$

Substituting eq. (147) in eq. (39) for single-site response we have

$$d\Phi|_{\mathbf{r}_a} = \sum_{q=1}^{N_Q}\tau_q ds_q, \tag{149}$$

$$\tau_q = \sum_{p\in Polars}\left\langle \Psi_q(\mathbf{r})\mathbf{G}^{ej}(\mathbf{j}_p),\ \mathbf{G}^e\left(\left(\frac{\partial\Phi}{\partial\mathbf{E}_p},\ \frac{\partial\Phi}{\partial\mathbf{H}_p}\right)\delta_{\mathbf{r}_a}\right)\right\rangle, \qquad q = 1, \dots, N_Q. \tag{150}$$

Modifications of eq. (150) for multisite responses and responses with spatial derivatives are obvious. Further, substituting eq. (147) in eq. (44) we obtain

$$d\phi_d = 2\mathrm{Re}\sum_{q=1}^{N_Q}\nu_q ds_q, \tag{151}$$

where

$$\nu_q = \sum_{\omega\in\Omega}\sum_{p\in Polars}\left\langle\Psi_q(\mathbf{r})\mathbf{G}^{ej}(\mathbf{j}_p),\ \ \mathbf{G}^e\left(\mathbf{J}_p^E, \mathbf{J}_p^H\right)\right\rangle, \qquad q = 1, \dots, N_Q, \tag{152}$$

$$\mathbf{J}_p^E = \sum_{k\in Resps}\sum_{a\in Sites}\left(\Phi_a^{(k)} - \Phi_a^{(k),\,exp}\right)^* D_a^{(k)}\frac{\partial\Phi_a^{(k)}}{\partial\mathbf{E}_p}\delta_{\mathbf{r}_a}, \qquad p \in Polars, \tag{153}$$

$$\mathbf{J}_p^H = \sum_{k\in Resps}\sum_{a\in Sites}\left(\Phi_a^{(k)} - \Phi_a^{(k),\,exp}\right)^* D_a^{(k)}\frac{\partial\Phi_a^{(k)}}{\partial\mathbf{H}_p}\delta_{\mathbf{r}_a}. \tag{154}$$

Here we remark that eqs (147)–(154) can be easily adapted to the anisotropic case by applying the modifications described in Section 7.1.

7.3 Non-holomorphic responses

So far we implicitly assumed that responses (see eq. 16) are complex differentiable (i.e. holomorphic) functions of conductivity σ. However often researchers work with non-holomorphic response functions, for example, in MT case with apparent resistivity

$$\rho_{xy}^{app} = \frac{|Z_{xy}|^2}{\omega\mu}, \tag{155}$$

or impedance phase

$$\varphi_{xy} = \arg Z_{xy}, \tag{156}$$

where Z_{xy} is as in eq. (17). But non-holomorphic responses can be derived from holomorphic responses with the use of real part operation, $\mathrm{Re}(\cdot)$, or imaginary part operation, $\mathrm{Im}(\cdot)$

$$\Psi = \Pi\,(\mathrm{Re}\,\Phi)\,, \tag{157}$$

$$\Theta = \Gamma\,(\mathrm{Im}\,\Phi)\,, \tag{158}$$

For example, apparent resistivity can be represented in the following form

$$\rho_{xy}^{app} = \frac{\exp(2\mathrm{Re}\,\Phi)}{\omega\mu}, \tag{159}$$

whereas impedance phase can be written as

$$\varphi_{xy} = \mathrm{Im}\,\Phi, \tag{160}$$

where $\Phi = \ln Z_{xy}$ is a holomorphic response function.

Let us now derive the differential of Ψ

$$d\Psi = d(\Pi\,(\mathrm{Re}\,\Phi)) = \Pi'\,(\mathrm{Re}\,\Phi)\,d(\mathrm{Re}\,\Phi) = \Pi'\,(\mathrm{Re}\,\Phi)\,\mathrm{Re}\,(d\Phi)\,, \tag{161}$$

where $d\Phi$ is as in eq. (39) (or in eqs E6 and F3). Here we use that operation $\mathrm{Re}(\cdot)$ is real linear function, hence $d(\mathrm{Re}\,\Phi) = \mathrm{Re}\,(d\Phi)$. From eq. (161) the generalization of eq. (39) is straightforward

$$d\Psi = \Pi'(\mathrm{Re}\,\Phi)\cdot\mathrm{Re}\sum_{p\in Polars}\left\langle \mathbf{G}^e\left(\frac{\partial\Phi}{\partial\mathbf{E}_p}\delta_{\mathbf{r}_a},\ \frac{\partial\Phi}{\partial\mathbf{H}_p}\delta_{\mathbf{r}_a}\right),\ \ d\sigma\,\mathbf{G}^{ej}(\mathbf{j}_p)\right\rangle. \tag{162}$$

In a similar way we obtain for $d\Theta$

$$d\Theta = \Gamma'(\mathrm{Im}\,\Phi)\cdot\mathrm{Im}\sum_{p\in Polars}\left\langle \mathbf{G}^e\left(\frac{\partial\Phi}{\partial\mathbf{E}_p}\delta_{\mathbf{r}_a},\ \frac{\partial\Phi}{\partial\mathbf{H}_p}\delta_{\mathbf{r}_a}\right),\ \ d\sigma\,\mathbf{G}^{ej}(\mathbf{j}_p)\right\rangle. \tag{163}$$

In our example eqs (162) and (163) read as follows

$$d\rho_{xy}^{app} = 2\rho_{xy}^{app}\cdot\mathrm{Re}\left(\frac{dZ_{xy}}{Z_{xy}}\right), \tag{164}$$

$$d\varphi_{xy} = \mathrm{Im}\left(\frac{dZ_{xy}}{Z_{xy}}\right), \tag{165}$$

where dZ_{xy} is calculated using eqs (67)–(70).

8 CONCLUSIONS

We have presented a general formalism for the efficient calculation of the derivatives of EM frequency-domain responses and the derivatives of the misfit with respect to variations of 3-D isotropic/anisotropic conductivity. The formalism works with single-site responses, multisite responses and responses that include spatial derivatives of the EM field. The corresponding responses may be either holomorphic or non-holomorphic. The formalism also allows for various types of parametrizations of the 3-D conductivity distribution. Using this formalism one can readily obtain appropriate formulae for the specific sounding methods. To illustrate the concept we have provided such formulae for a number of EM techniques: GDS, conventional and generalized magnetotellurics, MV method, HGS and a method that combines HGS with GDS. We have also shown how the developed formalism can be adapted for the inversion of multisite responses—horizontal magnetic and electric tensors.

ACKNOWLEDGMENTS

We wish to thank Chris Finlay who helped us to improve the English presentation of this paper. We also thank two anonymous reviewers for helpful suggestions on how to improve the manuscript. This work has been supported in part by European Space Agency through contract No. 20944/07/NL/JA and by Russian Foundation for Basic Research under grant No. 09-05-01071-a.

REFERENCES

Avdeev, D.B. & Avdeeva, A.D., 2009. 3D magnetotelluric inversion using a limited-memory quasi-Newton optimization, *Geophysics,* **74,** 45–57.

Banks, R., 1969. Geomagnetic variations and the electrical conductivity of the upper mantle, *Geophys. J. R. astr. Soc.,* **17,** 457–487.

Dmitriev, V.I. & Berdichevsky, M.N., 2002. A generalized impedance model, *Phys. Solid Earth,* **38,** 897–903.

Dorn, O., Bertete-Aquirre, H., Berryman, J.G. & Papanicolaou, G.C., 1999. A nonlinear inversion method for 3-D electromagnetic imaging using adjoint fields, *Inverse Problems,* **15,** 1523–1558.

Haber, E., 2005. Quasi-Newton methods for large-scale electromagnetic inverse problems, *Inverse Problems,* **21,** 305–323.

Haber, E., Oldenburg, D. & Shekhtman, R., 2007. Inversion of 3D time domain electromagnetic data, *Geophys. J. Int.,* **171,** 550–564.

Kelbert, A., Egbert, G. & Schultz, A., 2008. Non-linear conjugate gradient inversion for global EM induction: resolution studies, *Geophys. J. Int.,* **173,** 365–381.

Mackie, R.L. & Madden, T., 1993, Three-dimensional magnetotelluric inversion using conjugate gradients, *Geophys. J. Int.,* **115,** 215–229.

McGillivray, P.R., Oldenburg, D.W., Ellis, R.G. & Habashy, T.M., 1994. Calculation of sensitivities for the frequency-domain electromagnetic problem, *Geophys. J. Int.,* **116,** 1–4.

Newman, G.A. & Alumbaugh, D.L., 2000. Three-dimensional magnetotelluric inversion using non-linear conjugate gradients, *Geophys. J. Int.,* **140,** 410–424.

Newman, G.A. & Commer, M., 2005. New advances in three-dimensional transient electromagnetic inversion, *Geophys. J. Int.,* **160,** 5–32.

Nocedal, J. & Wright, S.J., 2006. *Numerical Optimization*. Springer, New York.

Olsen, N., 1998. The electrical conductivity of the mantle beneath Europe derived from C-responses from 3 to 720 hr, *Geophys. J. Int.,* **133,** 298–308.

Pellerin, L., Johnston, J.M. & Hohmann, G.W., 1993. Three-dimensional inversion of electromagnetic data, in *63th Ann. Int. Meeting SEG Expanded Abstracts*, Washington, DC, Soc. Exploration Geophysics, 360–363.

Rodi, W. & Mackie, R.L., 2000. Nonlinear conjugate gradients algorithm for 2-D magnetotelluric inversion, *Geophysics,* **66,** 174–187.

Schmucker, U., 1970. Anomalies of geomagnetic variations in the southwestern United States, *Bull. Scripps Inst. Ocean., Univ. Calif.,* **13,** 1–165.

Schmucker, U., 2003. Horizontal special gradient sounding and geomagnetic sounding in the period range of daily variation, *Protokoll uber das Kolloquium electromagnetische Tiefenforschung,* **20,** 228–237.

Semenov, V.Yu., Vozár, J. & Shuman, V., 2007. A new approach to gradient geomagnetic sounding, *Izvestiya, Phys. Solid Earth,* **43**(7), 592–596.

Zhdanov, M., 2002. Geophysical inverse theory and regularization problems, in *Methods in Geochemistry and Geophysics,* Vol. **36**, Elsevier, Amsterdam.

APPENDIX A: RECIPROCITY OF GREEN'S FUNCTIONS $\mathbf{G}^{ej}$, $\mathbf{G}^{eh}$ AND $\mathbf{G}^{hj}$

In this Appendix, we show that the Green's functions $\mathbf{G}^{ej}$, $\mathbf{G}^{eh}$ and $\mathbf{G}^{hj}$ defined in eqs (2), (3) and (6) satisfy the following relations

$$\langle \mathbf{G}^{ej}(\mathbf{a}),\ \ \mathbf{b}\rangle = \langle \mathbf{a},\ \ \mathbf{G}^{ej}(\mathbf{b})\rangle\ , \tag{A1}$$

$$\langle \mathbf{G}^{eh}(\mathbf{a}),\ \ \mathbf{b}\rangle = \langle \mathbf{a},\ \ \mathbf{G}^{hj}(\mathbf{b})\rangle\ , \tag{A2}$$

for any vector fields $\mathbf{a} = \mathbf{a}(\mathbf{r})$ and $\mathbf{b} = \mathbf{b}(\mathbf{r})$. Angle brackets used here are defined in eq. (31). First, let us obtain—from Maxwell's equations (2)—the second-order differential equation for the electric field. Substituting the second equation of (2) into the first equation of (2) we have

$$\nabla \times \left(\frac{\nabla \times \mathbf{E}}{i\omega\mu} \right) - \sigma \mathbf{E} = \mathbf{j}^{imp}. \tag{A3}$$

Let $\mathbf{A} = \mathbf{G}^{ej}(\mathbf{a})$ and $\mathbf{B} = \mathbf{G}^{ej}(\mathbf{b})$, that is,

$$\nabla \times \left(\frac{\nabla \times \mathbf{A}}{i\omega\mu} \right) - \sigma \mathbf{A} = \mathbf{a}, \tag{A4}$$

and

$$\nabla \times \left(\frac{\nabla \times \mathbf{B}}{i\omega\mu} \right) - \sigma \mathbf{B} = \mathbf{b}. \tag{A5}$$

Using reciprocity of $\nabla\times$ operator, that is,

$$\langle \nabla \times \mathbf{a},\ \ \mathbf{b}\rangle = \langle \mathbf{a},\ \ \nabla \times \mathbf{b}\rangle\,, \tag{A6}$$

we obtain eq. (A1) as a result of the following sequence of equalities

$$\begin{aligned}\langle \mathbf{G}^{ej}(\mathbf{a}),\ \ \mathbf{b}\rangle &= \left\langle \mathbf{A},\ \ \nabla \times \left(\frac{\nabla \times \mathbf{B}}{i\omega\mu} \right) - \sigma \mathbf{B} \right\rangle = \left\langle \mathbf{A},\ \ \nabla \times \left(\frac{\nabla \times \mathbf{B}}{i\omega\mu} \right) \right\rangle - \langle \mathbf{A},\ \ \sigma \mathbf{B}\rangle \\ &= \left\langle \nabla \times \left(\frac{\nabla \times \mathbf{A}}{i\omega\mu} \right),\ \ \mathbf{B} \right\rangle - \langle \sigma \mathbf{A},\ \ \mathbf{B}\rangle = \left\langle \nabla \times \left(\frac{\nabla \times \mathbf{A}}{i\omega\mu} \right) - \sigma \mathbf{A},\ \ \mathbf{B} \right\rangle = \langle \mathbf{a},\ \ \mathbf{G}^{ej}(\mathbf{b})\rangle.\end{aligned} \tag{A7}$$

Note that reciprocity (A6) of the $\nabla\times$ operator follows from the integration by parts over the whole Euclidean space $\mathbb{R}^3$ and from the condition at the infinity, see eq. (2). Similar calculations prove eq. (A2).

APPENDIX B: PREDICTED RESPONSE FUNCTIONS AS A COUPLING MATRIX

Let (k) in (16) be a double superscript $(k) = (\alpha, \beta)$, $\alpha \in Dim_g$, $\beta \in Dim_p$, where $Dim_g = \{\alpha_1, \ldots, \alpha_{N_g}\}$ and $Dim_p = \{\beta_1, \ldots, \beta_{N_p}\}$ are enumerator sets to indicate the EM field components or their multisite linear combinations or their spatial derivatives (see Appendices E and F). As an example, for the combined 'GDS+HGS' method (see Section 6.8) these sets are

$$Dim_g = \{r\}, \quad Dim_p = \{u, \vartheta, \varphi\}, \quad N_g = 1, \quad N_p = 3, \tag{B1}$$

where $r,\ \vartheta,\ \varphi$ indicate the coordinates of spherical coordinate system and u indicates the angular part of the divergence (see eq. 117). For conventional (see Section 6.2) and generalized (see Section 6.3) MT methods these sets are

$$Dim_g = Dim_p = \{x, y\}, \quad N_g = N_p = 2, \tag{B2}$$

and

$$Dim_g = \{x, y\}, \quad Dim_p = \{x, y, z\}, \quad N_g = 2,\ N_p = 3. \tag{B3}$$

respectively.

Before introducing the general definition for coupling matrix let us consider again conventional MT case as an example. Here the impedance tensor is a 2×2 MT matrix

$$\mathbf{C} = \mathbf{Z} = \begin{pmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{pmatrix}, \tag{B4}$$

which connects horizontal electric field components with horizontal magnetic field components (see eq. 17). We can consider this tensor as a coupling matrix between horizontal electric and magnetic field. The coupling matrix elements can then be calculated as follows

$$\mathbf{C} = \begin{pmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{pmatrix} = \mathbf{RQ}, \qquad \mathbf{R} = \begin{pmatrix} E_{x1} & E_{x2} \\ E_{y1} & E_{y2} \end{pmatrix}, \qquad \mathbf{Q} = \mathbf{S}^{-1}, \qquad \mathbf{S} = \begin{pmatrix} H_{x1} & H_{x2} \\ H_{y1} & H_{y2} \end{pmatrix}, \tag{B5}$$

where $E_{1x},\ E_{1y},\ E_{2x},\ E_{2y}$ and $H_{1x},\ H_{1y},\ H_{2x},\ H_{2y}$ are, respectively, the horizontal components of electric and magnetic fields due to the two sources that are two primary plane waves of different polarization normally incident on the air-ground interface.

In the general case we introduce two sets of (linear differential operators of) EM field components

$$R_\alpha, \qquad \alpha \in Dim_g, \tag{B6}$$

and

$$S_\beta, \qquad \beta \in Dim_p, \tag{B7}$$

and search for the coupling coefficients

$$C_{\alpha\beta}, \qquad \alpha \in Dim_g,\ \beta \in Dim_p, \tag{B8}$$

to satisfy equation

$$R_\alpha = \sum_{\beta \in Dim_p} C_{\alpha\beta} S_\beta, \qquad \alpha \in Dim_g, \tag{B9}$$

for any polarization of impressed sources (7)–(8) and their linear combinations. The grounds for eq. (B9) is the linear independence of the impressed sources (7)–(8), linearity of Maxwell's equations (2), and linear independence of set (B7). To calculate coupling coefficients (B8) we use linear algebra; particularly, we constitute two matrices

$$\mathbf{R} = \begin{pmatrix} R_{\alpha_1}(\mathbf{j}_1) & \cdots & R_{\alpha_1}(\mathbf{j}_{N_p}) \\ \vdots & & \vdots \\ R_{\alpha_{N_d}}(\mathbf{j}_1) & \cdots & R_{\alpha_{N_d}}(\mathbf{j}_{N_p}) \end{pmatrix}, \tag{B10}$$

and

$$\mathbf{S} = \begin{pmatrix} S_{\beta_1}(\mathbf{j}_1) & \cdots & S_{\beta_1}(\mathbf{j}_{N_p}) \\ \vdots & & \vdots \\ S_{\beta_{N_p}}(\mathbf{j}_1) & \cdots & S_{\beta_{N_p}}(\mathbf{j}_{N_p}) \end{pmatrix}, \tag{B11}$$

and deduce that identity (B9) is equivalent to matrix equation

$$\mathbf{R} = \mathbf{CS}, \tag{B12}$$

where

$$\mathbf{C} = \begin{pmatrix} C_{\alpha_1\beta_1} & \cdots & C_{\alpha_1\beta_{N_p}} \\ \vdots & & \vdots \\ C_{\alpha_{N_d}\beta_1} & \cdots & C_{\alpha_{N_d}\beta_{N_p}} \end{pmatrix}, \tag{B13}$$

is the desired coupling matrix. The solution to matrix eq. (B12) can be found if $\det \mathbf{S} \neq 0$. Then the inverse matrix

$$\mathbf{Q} = \mathbf{S}^{-1}, \tag{B14}$$

can be represented in the form of

$$\mathbf{S}^{-1} = \frac{\mathrm{adj}\,\mathbf{S}}{\det \mathbf{S}}, \tag{B15}$$

where the elements of adjugate matrix, adj $\mathbf{S} = \mathbf{A}$, are determined as follows

$$A_{p\beta} = (-1)^{p+[\beta]} \det(\mathrm{del}_{\beta p}(\mathbf{S})),\ \ p \in Polars,\ \ \beta \in Dim_b. \tag{B16}$$

Here $\mathrm{del}_{\beta p}(\mathbf{S})$ is $(N_p - 1) \times \left(N_p - 1\right)$ is the submatrix of $\mathbf{S}$ which remains after deletion of $[\beta]$th row and pth column of matrix $\mathbf{S}$. Here $[\beta]$ is a serial number of β within ordered set Dim_p. As an example, in the conventional MT case

$$\mathbf{Q} = \begin{pmatrix} Q_{1x} & Q_{1y} \\ Q_{2x} & Q_{2y} \end{pmatrix} = \mathbf{S}^{-1} = \frac{1}{D}\begin{pmatrix} H_{y2} & -H_{x2} \\ -H_{y1} & H_{x1} \end{pmatrix}, \qquad D = \begin{vmatrix} H_{x1} & H_{x2} \\ H_{y1} & H_{y2} \end{vmatrix}. \tag{B17}$$

APPENDIX C: DIFFERENTIAL OF COUPLING MATRIX

In this Appendix, we present explicit formulae for calculating the differential of coupling matrix $\mathbf{C}$ with respect to $\mathbf{R}$ and $\mathbf{S}$. These formulae are exploited in Section 6. First, note that eq. (B14) means that

$$\mathbf{QS} = \mathbf{1}, \tag{C1}$$

where $\mathbf{1}$ is $N_p \times N_p$ identity matrix. Thus the solution to eq. (B12) can be written as

$$\mathbf{C} = \mathbf{RQ}, \tag{C2}$$

where matrix $\mathbf{Q}$ is expressed in terms of $\mathbf{S}$ using eqs (B15)–(B16). This in particular means that the differential of $\mathbf{Q}$ can be expressed as a linear combination of the differential of $\mathbf{R}$ and $\mathbf{S}$

$$dC_{\xi\eta} = \sum_{\substack{\alpha\in Dim_g \\ p\in Polars}} \frac{\partial C_{\xi\eta}}{\partial R_{\alpha p}} dR_{\alpha p} + \sum_{\substack{\beta\in Dim_p \\ p\in Polars}} \frac{\partial C_{\xi\eta}}{\partial S_{\beta p}} dS_{\beta p}, \quad \forall \xi \in Dim_g, \quad \eta \in Dim_p, \tag{C3}$$

where

$$R_{\alpha p} = R_\alpha(\mathbf{j}_p), \qquad S_{\beta p} = S_\beta(\mathbf{j}_p), \tag{C4}$$

Let us now explicitly calculate the differential of coupling matrix $\mathbf{C}$ with respect to $\mathbf{R}$ and $\mathbf{S}$. Let us first calculate the differential of the inverse matrix

$$d(\mathbf{S}^{-1}) = -\mathbf{S}^{-1}\,(d\mathbf{S})\,\mathbf{S}^{-1}. \tag{C5}$$

Eq. (C5) follows from Leibniz's law and eq. (C1). Indeed

$$(d\mathbf{Q})\,\mathbf{S} + \mathbf{Q}\,(d\mathbf{S}) = \mathbf{0}. \tag{C6}$$

Further, calculating the differential of the coupling matrix from eqs (B14), (C2), (C5) and again from Leibniz's law we obtain

$$d\mathbf{C} = (d\mathbf{R})\,\mathbf{Q} - \mathbf{C}\,(d\mathbf{S})\,\mathbf{Q}, \tag{C7}$$

that is

$$dC_{\xi\eta} = \sum_{p\in Polars} (dR_{\xi p})Q_{p\eta} - \sum_{\substack{p\in Polars \\ \beta\in Dim_p}} C_{\xi\beta}(dS_{\beta p})Q_{p\eta}, \qquad \forall \xi \in Dim_g, \quad \eta \in Dim_p. \tag{C8}$$

Comparing eqs (C8) and (C3) we finally derive the coefficients of the coupling matrix sensitivity to $\mathbf{R}$ and $\mathbf{S}$

$$\frac{\partial C_{\xi\eta}}{\partial R_{\alpha p}} = \delta_{\xi\alpha} Q_{p\eta}, \qquad \frac{\partial C_{\xi\eta}}{\partial S_{\beta p}} = -C_{\xi\beta} Q_{p\eta}, \qquad \forall \xi, \alpha \in Dim_g, \quad \forall \beta, \eta \in Dim_p, \quad \forall p \in Polars. \tag{C9}$$

APPENDIX D: DIFFERENTIAL OF THE QUADRATIC MISFIT OF THE RESPONSE FUNCTIONS

In this Appendix, we derive eq. (43) that is the differential of the quadratic misfit defined in eq. (21). First, we write eq. (21) in the following form

$$\begin{cases} \phi_d\,(m) = \sum\limits_{\omega\in\Omega} \sum\limits_{k\in Resps} \sum\limits_{a\in Sites} D_a^{(k)}(\omega) \cdot |u_{a,k}|^2, \\ \quad u_{a,k} = \Phi_a^{(k)}(\mathbf{m}, \omega) - \Phi_a^{(k),\,exp}(\omega). \end{cases} \tag{D1}$$

Second, we write the differential of $|u_{a,k}|^2$ as

$$d|u_{a,k}|^2 = d\left(u_{a,k}u_{a,k}^*\right) = u_{a,k}^* du_{a,k} + u_{a,k} du_{a,k}^* = u_{a,k}^* du_{a,k} + \left(u_{a,k}^* du_{a,k}\right)^* = 2\mathrm{Re}\left(u_{a,k}^*\, du_{a,k}\right). \tag{D2}$$

Using the latter equation we finally get for $d\phi_d$

$$d\phi_d(\mathbf{m}) = \sum_{\omega\in\Omega}\sum_{k\in Resps}\sum_{a\in Sites} D_a^{(k)}(\omega)\, d|u_{a,k}|^2 = 2\mathrm{Re}\sum_{\omega\in\Omega}\sum_{k\in Resps}\sum_{a\in Sites} u_{a,k}^*\, D_a^{(k)}(\omega)\, du_{a,k}. \tag{D3}$$

Here we also use the fact that $D_a^{(k)}(\omega)$ does not depend on the model vector $\mathbf{m}$.

APPENDIX E: FORMALISM FOR THE MULTISITE RESPONSE FUNCTIONS

This Appendix is relevant to Sections 6.5 and 6.6 and discusses differentials of multisite responses and the differential of their misfit. We write multisite responses as follows (*cf*. eq. 16 for single-site responses)

$$\Phi^{(n)} = \Phi^{(n)}\left(\{\mathbf{E}_p(\mathbf{r}_f), \mathbf{H}_p(\mathbf{r}_f)\}_{p\in Polars,\ f\in Sites}\right). \tag{E1}$$

Here $n \in Samples$, where $\{\Phi^{(n)}\}_{n\in Samples}$ is the set of the response samples. Superscript $(n) = (k, s)$ can be regarded as a pair of an index $k \in Resps$, and an index s. The index k identifies the specific response function, whereas the index s defines the combination of the observation sites. For example, an element of the horizontal magnetic tensor, $M_{\xi\eta}^{ab}$ (see eq. 94), depends on ξ, η, a and b. Hence here the superscript (n) is equal to $(k, s) = (\xi, \eta, a, b)$, where $(k) = (\xi, \eta)$ specifies a response type ('*xx*', '*xy*', '*yx*' or '*yy*'), and $s = (a, b)$, $a, b \in Sites$ determines a pair of the observation sites.

We rewrite the data misfit $\phi_d(\mathbf{m})$ (*cf*. eq. 21 for single-site misfit) as follows

$$\phi_d(\mathbf{m}) = \sum_{n\in Samples}\sum_{\omega\in\Omega} D^{(n)}\, |\Phi^{(n)} - \Phi^{(n),\,exp}|^2. \tag{E2}$$

Here $\Phi^{(n)} = \Phi^{(n)}(\mathbf{m}, \omega, \{\mathbf{r}_f\}_{f\in Sites})$ and $\Phi^{(n),\,exp} = \Phi^{(n),\,exp}(\omega, \{\mathbf{r}_f\}_{f\in Sites})$ are, respectively, the predicted and observed values of the multisite response functions, and $D^{(n)} = D^{(n)}(\omega, \{\mathbf{r}_f\}_{f\in Sites})$ are the inverses of squared uncertainties of the observed responses.

For the multisite case, eqs (22), (29), (30), (39) and (40)–(48) are transformed into the following equations

$$d\Phi^{(n)} = \sum_{p\in Polars}\left(d_{\mathbf{E}_p}\Phi^{(n)} + d_{\mathbf{H}_p}\Phi^{(n)}\right), \tag{E3}$$

$$d_{\mathbf{E}_p}\Phi^{(n)} = \frac{\partial\Phi^{(n)}}{\partial\mathbf{E}_p}\cdot d\mathbf{E}_p = \left\langle \mathbf{G}^{ej}\left(\sum_{f\in Sites}\frac{\partial\Phi^{(n)}}{\partial\mathbf{E}_p(\mathbf{r}_f)}\delta_{\mathbf{r}_f}\right),\quad d\sigma\ \mathbf{G}^{ej}(\mathbf{j}_p)\right\rangle, \tag{E4}$$

$$d_{\mathbf{H}_p}\Phi^{(n)} = \frac{\partial\Phi^{(n)}}{\partial\mathbf{H}_p}\cdot d\mathbf{H}_p = \left\langle \mathbf{G}^{eh}\left(\sum_{f\in Sites}\frac{\partial\Phi^{(n)}}{\partial\mathbf{H}_p(\mathbf{r}_f)}\delta_{\mathbf{r}_f}\right),\quad d\sigma\ \mathbf{G}^{ej}(\mathbf{j}_p)\right\rangle, \tag{E5}$$

$$d\Phi^{(n)} = \sum_{p\in Polars}\left\langle \mathbf{G}^{e}\left(\sum_{f\in Sites}\left(\frac{\partial\Phi^{(n)}}{\partial\mathbf{E}_p(\mathbf{r}_f)}, \frac{\partial\Phi^{(n)}}{\partial\mathbf{H}_p(\mathbf{r}_f)}\right)\delta_{\mathbf{r}_f}\right),\quad d\sigma\ \mathbf{G}^{ej}(\mathbf{j}_p)\right\rangle, \tag{E6}$$

$$d\Phi^{(n)} = \sum_{m\in Model}\kappa_m^{(n)} d\sigma_m, \tag{E7}$$

$$\kappa_m^{(n)} = \sum_{p\in Polars}\int_{V_m}\mathbf{G}^{ej}(\mathbf{j}_p)\cdot\mathbf{G}^{e}\left(\sum_{f\in Sites}\left(\frac{\partial\Phi^{(n)}}{\partial\mathbf{E}_p(\mathbf{r}_f)}, \frac{\partial\Phi^{(n)}}{\partial\mathbf{H}_p(\mathbf{r}_f)}\right)\delta_{\mathbf{r}_f}\right)dv(\mathbf{r}), \tag{E8}$$

$$\frac{\partial\Phi^{(n)}}{\partial\sigma_m} = \kappa_m^{(n)},\quad n\in Samples,\quad m\in Model. \tag{E9}$$

$$d\phi_d = 2\mathrm{Re}\sum_{\omega\in\Omega}\sum_{n\in Samples}(\Phi^{(n)} - \Phi^{(n),exp})^* D^{(n)}\ d\Phi^{(k)}, \tag{E10}$$

$$d\phi_d = 2\mathrm{Re}\sum_{\omega\in\Omega}\sum_{p\in Polars}\left\langle \mathbf{G}^{e}\left(\mathbf{J}_p^{E}(\omega), \mathbf{J}_p^{H}(\omega)\right),\ d\sigma\ \mathbf{G}^{ej}(\mathbf{j}_p)\right\rangle, \tag{E11}$$

$$\mathbf{J}_p^{E}(\omega) = \sum_{n\in Samples}(\Phi^{(n)} - \Phi^{(n),exp})^* D^{(n)}\sum_{f\in Sites}\frac{\partial\Phi^{(n)}}{\partial\mathbf{E}_p(\mathbf{r}_f)}\delta_{\mathbf{r}_f}, \tag{E12}$$

$$\mathbf{J}_p^H(\omega) = \sum_{n \in Samples} (\Phi^{(n)} - \Phi^{(n),exp})^* D^{(n)} \sum_{f \in Sites} \frac{\partial \Phi^{(n)}}{\partial \mathbf{H}_p(\mathbf{r}_f)} \delta_{\mathbf{r}_f}, \tag{E13}$$

$$d\phi_d = 2\mathrm{Re} \sum_{m \in Model} \lambda_m \, d\sigma_m, \tag{E14}$$

$$\lambda_m = \sum_{\omega \in \Omega} \sum_{p \in Polars} \int_{V_m} \mathbf{G}^{ej}(\mathbf{j}_p(\omega)) \cdot \mathbf{G}^e \left(\mathbf{J}_p^E(\omega), \mathbf{J}_p^H(\omega)\right) dv\,(\mathbf{r}). \tag{E15}$$

Finally note that eqs (49)–(53) hold for this multisite case.

Naturally, if the summation over f contains only one term then the equations for the multisite case degenerate to eqs (16)–(53) for the single-site case.

APPENDIX F: FORMALISM FOR THE MULTISITE RESPONSE FUNCTIONS THAT INCLUDE SPATIAL DERIVATIVES

This Appendix is relevant to Sections 6.7 and 6.8. This case is an extension of that discussed in Appendix E. The distinction is that now multisite responses depend not only on electric and magnetic fields but also on their spatial derivatives. We write the responses for this case as follows

$$\begin{cases} \Phi^{(n)} = \Phi^{(n)} \left(\{u_{fp}\}_{f \in Sites,\ p \in Polars}\right), & n \in Samples, \\ u_{fp} = L_f(\mathbf{E}_p, \mathbf{H}_p), \qquad f \in Sites, & p \in Polars, \end{cases} \tag{F1}$$

where *Samples* is defined in Appendix E. Here L_f is a spatial linear differential operator at the observation site $\mathbf{r}_f$, $f \in Sites$

$$L_f(\mathbf{E}, \mathbf{H}) = \sum_{\substack{m=(m_1,m_2,m_3) \\ \alpha=\xi,\,\gamma,\,\zeta}} \Lambda_{f\xi}^{em}(\mathbf{r}) \frac{\partial^{|m|} E_\alpha(\mathbf{r})}{\partial^{m_1}\chi\ \partial^{m_2}\gamma\ \partial^{m_3}\zeta}\bigg|_{\mathbf{r}=\mathbf{r}_f} + \sum_{\substack{m=(m_1,m_2,m_3) \\ \alpha=\xi,\,\gamma,\,\zeta}} \Lambda_{f\xi}^{hm}(\mathbf{r}) \frac{\partial^{|m|} H_\alpha(\mathbf{r})}{\partial^{m_1}\chi\ \partial^{m_2}\gamma\ \partial^{m_3}\zeta}\bigg|_{\mathbf{r}=\mathbf{r}_f}, \tag{F2}$$

where $m = (m_1, m_2, m_3)$, m_j are non-negative integers; $|m| = m_1 + m_2 + m_3$; $\Lambda_{f\xi}^{em}(\mathbf{r})$ and $\Lambda_{f\xi}^{hm}(\mathbf{r})$ are coefficients of the operator; $\mathbf{r} = \mathbf{r}(\chi, \gamma, \zeta)$ are any coordinates. The misfit for this case has a form of eq. (E2).

Eqs (E3)–(E15) for this case are modified into the following equations

$$d\Phi^{(n)} = \frac{\partial \Phi^{(n)}}{\partial u} \cdot du = \sum_{\substack{f \in Sites \\ p \in Polars}} \frac{\partial \Phi^{(n)}}{\partial u_{fp}} du_{fp} = \sum_{p \in Polars} \left\langle \mathbf{G}^e \left(\sum_{f \in Sites} \frac{\partial \Phi^{(n)}}{\partial u_{fp}} L_f^* \delta_{\mathbf{r}_f} \right), \quad d\sigma\ \mathbf{G}^{ej}(\mathbf{j}_p) \right\rangle, \tag{F3}$$

$$d\Phi^{(n)} = \sum \kappa_m^{(n)} d\sigma_m, \qquad n \in Samples, \tag{F4}$$

$$\kappa_m^{(n)} = \sum_{p \in Polars} \int_{V_m} \mathbf{G}^{ej}(\mathbf{j}_p) \cdot \mathbf{G}^e \left(\sum_{f \in Sites} \frac{\partial \Phi^{(n)}}{\partial u_{fp}} L_f^* \delta_{\mathbf{r}_f} \right) dv\,(\mathbf{r}), \qquad m \in Model, \tag{F5}$$

$$\frac{\partial \Phi^{(n)}}{\partial \sigma_m} = \kappa_m^{(n)}, \qquad n \in Samples, \quad m \in Model, \tag{F6}$$

$$d\phi_d = 2\mathrm{Re} \sum_{\omega \in \Omega} \sum_{n \in Samples} (\Phi^{(n)} - \Phi^{(n),exp})^* D^{(n)}\ d\Phi^{(n)}, \tag{F7}$$

$$d\phi_d = 2\mathrm{Re} \sum_{\omega \in \Omega} \sum_{p \in Polars} \left\langle \mathbf{G}^e \left(\mathbf{J}_p^u(\omega)\right), \ d\sigma\ \mathbf{G}^{ej}(\mathbf{j}_p) \right\rangle, \tag{F8}$$

$$\mathbf{J}_p^u(\omega) = \sum_{n \in Samples} (\Phi^{(n)} - \Phi^{(n),exp})^* D^{(n)} \sum_{f \in Sites} \frac{\partial \Phi^{(n)}}{\partial u_{fp}} L_f^* \delta_{\mathbf{r}_f}, \qquad p \in Polars, \tag{F9}$$

$$d\phi_d = 2\mathrm{Re} \sum_{m \in Model} \lambda_m \, d\sigma_m, \tag{F10}$$

$$\lambda_m = \sum_{\omega \in \Omega} \sum_{p \in Polars} \int_{V_m} \mathbf{G}^{ej}(\mathbf{j}_p(\omega)) \cdot \mathbf{G}^e \left(\mathbf{J}_p^u(\omega)\right) dv(\mathbf{r}), \qquad m \in Model. \tag{F11}$$

Note that eqs (49)–(53) also hold for this case. The operator L_f^* in eqs (F3), (F5) and (F9) is a conjugate differential operator to L_f. In Cartesian coordinates L_f^* has the form

$$L_f^* U = \sum_{\substack{m=(m_1,m_2,m_3)\\ \xi=x,y,z}} (-1)^{|m|} \left(\mathbf{e}_\xi \frac{\partial^{|m|}\left(\Lambda_{f\xi}^{em}(\mathbf{r})U(\mathbf{r})\right)}{\partial^{m_1}x\,\partial^{m_2}y\,\partial^{m_3}z},\quad \mathbf{e}_\xi \frac{\partial^{|m|}\left(\Lambda_{f\xi}^{hm}(\mathbf{r})U(\mathbf{r})\right)}{\partial^{m_1}x\,\partial^{m_2}y\,\partial^{m_3}z} \right)\Bigg|_{\mathbf{r}=\mathbf{r}_f}, \qquad \forall U = U(\mathbf{r}), \qquad \mathbf{r} \in \mathbb{R}^3. \tag{F12}$$

The first term in the brackets corresponds to the electric field $\mathbf{E}$ whereas the second term corresponds to the magnetic field $\mathbf{H}$. In spherical coordinates expression for L_f^* is more complicated as it invokes the Lame coefficients. One example of a pair of conjugate operators L_f and L_f^* is div_τ and $-\nabla_\tau$:

$$\mathrm{div}_\tau^* = -\nabla_\tau, \tag{F13}$$

where τ stands for the angular part of the corresponding operators.

It is worthwhile to remark that eqs (F3)–(F11) reduce to eqs (E6)–(E15) provided that: (a) for each observation site $\mathbf{r}_{obs}$ we have exactly six elements f for that $\mathbf{r}_f = \mathbf{r}_{obs}$ in sequence $\{\mathbf{r}_f\}_{f\in Sites}$, and (b) differential operators $L_f(\mathbf{E},\mathbf{H})$ for these six elements have the form of $E_x|_{\mathbf{r}_{obs}}$, $E_y|_{\mathbf{r}_{obs}}$, $E_z|_{\mathbf{r}_{obs}}$, $H_x|_{\mathbf{r}_{obs}}$, $H_y|_{\mathbf{r}_{obs}}$, and $H_z|_{\mathbf{r}_{obs}}$.

Final note is that eq. (F1) can be regarded as a special case of eq. (E1) if the practical calculations of spatial derivatives are performed using small-size multisite arrays of electrodes or/and loops.